4 The Error Power as a Function of the Filter Coefficients

Error signal:

$$e(n) = d(n) - \sum_{i=0}^{N-1} h_i \, y(n-i) = d(n) - \boldsymbol{h}^{\mathrm{T}} \boldsymbol{y}(n)$$

Error power (for arbitrary filter vectors):

$$\mathrm{E}\{e^2(n)\} = \mathrm{E}\Big\{\big(d(n) - \boldsymbol{h}^{\mathrm{T}} \boldsymbol{y}(n)\big)\big(d(n) - \boldsymbol{h}^{\mathrm{T}} \boldsymbol{y}(n)\big)\Big\} = r_{dd}(0) - 2\,\boldsymbol{h}^{\mathrm{T}} \boldsymbol{r}_{yd}(0) + \boldsymbol{h}^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{h}$$

Error power (for the optimal filter vector):

$$\mathrm{E}\{e^2(n)\}\Big|_{\min} = r_{dd}(0) - 2\,\boldsymbol{h}_{\mathrm{opt}}^{\mathrm{T}} \boldsymbol{r}_{yd}(0) + \boldsymbol{h}_{\mathrm{opt}}^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{h}_{\mathrm{opt}}$$

.... *inserting* $\boldsymbol{h}_{\mathrm{opt}} = \boldsymbol{R}_{yy}^{-1} \boldsymbol{r}_{yd}(0)$

$$\mathrm{E}\{e^2(n)\}\Big|_{\min} = r_{dd}(0) - 2\left(\boldsymbol{R}_{yy}^{-1} \boldsymbol{r}_{yd}(0)\right)^{\mathrm{T}} \boldsymbol{r}_{yd}(0) + \left(\boldsymbol{R}_{yy}^{-1} \boldsymbol{r}_{yd}(0)\right)^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{R}_{yy}^{-1} \boldsymbol{r}_{yd}(0)$$

.... *inserting* $\left(\boldsymbol{R}_{yy}^{-1}\right)^{\mathrm{T}} = \boldsymbol{R}_{yy}^{-1}$

$$\mathrm{E}\{e^2(n)\}\Big|_{\min} = r_{dd}(0) - 2\,\boldsymbol{r}_{yd}^{\mathrm{T}}(0)\,\boldsymbol{R}_{yy}^{-1}\,\boldsymbol{r}_{yd}(0) + \boldsymbol{r}_{yd}^{\mathrm{T}}(0)\,\boldsymbol{R}_{yy}^{-1} \boldsymbol{R}_{yy}\,\boldsymbol{R}_{yy}^{-1}\,\boldsymbol{r}_{yd}(0) = r_{dd}(0) - \boldsymbol{r}_{yd}^{\mathrm{T}}(0)\,\boldsymbol{R}_{yy}^{-1}\,\boldsymbol{r}_{yd}(0)$$

Abbreviation:

$$E_{\min} = \mathrm{E}\{e^2(n)\}\Big|_{\min}$$

Error power:

$$\begin{aligned}
\mathrm{E}\{e^2(n)\} &= r_{dd}(0) - 2\,\boldsymbol{h}^{\mathrm{T}} \boldsymbol{r}_{yd}(0) + \boldsymbol{h}^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{h} \\
&= \overbrace{r_{dd}(0) - \Big(\boldsymbol{r}_{yd}^{\mathrm{T}}(0)\,\boldsymbol{R}_{yy}^{-1}\,\boldsymbol{r}_{yd}(0)}^{E_{\min}} \\
&\quad - \boldsymbol{r}_{yd}^{\mathrm{T}}(0)\,\boldsymbol{R}_{yy}^{-1}\,\boldsymbol{r}_{yd}(0)\Big) \\
&\quad - 2\,\boldsymbol{h}^{\mathrm{T}} \boldsymbol{r}_{yd}(0) + \boldsymbol{h}^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{h} \\
&= E_{\min} + \boldsymbol{r}_{yd}^{\mathrm{T}}(0)\,\boldsymbol{R}_{yy}^{-1}\,\boldsymbol{r}_{yd}(0) \\
&\quad - 2\,\boldsymbol{h}^{\mathrm{T}} \boldsymbol{r}_{yd}(0) + \boldsymbol{h}^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{h}
\end{aligned}$$

.... *inserting* $\boldsymbol{r}_{yd}(0) = \boldsymbol{R}_{yy}\,\boldsymbol{h}_{\mathrm{opt}}$

$$\begin{aligned}
\mathrm{E}\{e^2(n)\} &= E_{\min} + \boldsymbol{h}_{\mathrm{opt}}^{\mathrm{T}}\,\boldsymbol{R}_{yy}\,\boldsymbol{R}_{yy}^{-1}\,\boldsymbol{R}_{yy}\,\boldsymbol{h}_{\mathrm{opt}} \\
&\quad - 2\,\boldsymbol{h}^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{h}_{\mathrm{opt}} + \boldsymbol{h}^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{h} \\
&= E_{\min} + \boldsymbol{h}_{\mathrm{opt}}^{\mathrm{T}}\,\boldsymbol{R}_{yy}\,\boldsymbol{h}_{\mathrm{opt}} - \boldsymbol{h}^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{h}_{\mathrm{opt}} \\
&\quad - \boldsymbol{h}_{\mathrm{opt}}^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{h} + \boldsymbol{h}^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{h} \\
&= E_{\min} - \left(\boldsymbol{h} - \boldsymbol{h}_{\mathrm{opt}}\right)^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{h}_{\mathrm{opt}} \\
&\quad + \left(\boldsymbol{h} - \boldsymbol{h}_{\mathrm{opt}}\right)^{\mathrm{T}} \boldsymbol{R}_{yy}\,\boldsymbol{h}
\end{aligned}$$

$$\boxed{\mathrm{E}\{e^2(n)\} = E_{\min} + \left(\boldsymbol{h} - \boldsymbol{h}_{\mathrm{opt}}\right)^{\mathrm{T}} \boldsymbol{R}_{yy}\left(\boldsymbol{h} - \boldsymbol{h}_{\mathrm{opt}}\right)}$$

Examples:

see lecture slides