ADSP - Ex. 1

Problem 1 (relationship between continuous and discrete signals)

A complex-valued continuous-time signal $v_a(t)$ has the Fourier transform shown in figure 1. This signal is sampled to produce the sequence $v(n) = v_a(nT)$.

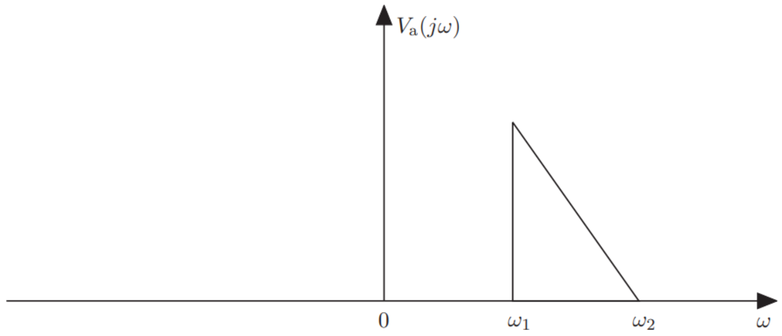


Figure 1: Fourier transform of $v_a(t)$

(a) Sketch the Fourier transform $V(e^{j\Omega})$ of the sequence $v(n)$ for $T = \frac{\pi}{\omega_2}$.

(b) What is the lowest sampling frequency that can be used without incurring any aliasing distortion, i.e. so that $v_a(t)$ can be recovered from $v(n)$?

a.) From Fourier transform ⟶ DTFT

$\underbrace{V(t)}_{\text{cont.}} \circ\!\!-\!\!\bullet \underbrace{V(j\omega)}_{\text{cont.}}$ $\underbrace{V(n)}_{\text{discrete}} \circ\!\!-\!\!\bullet \underbrace{V(e^{j\Omega})}_{\text{cont.}}$

↳ Sampling period : $T = \frac{\pi}{\omega_2}$

⇓

Sampling frequency: $f_s = \frac{1}{T} = \frac{\omega_2}{\pi}$

$\omega = 2\pi f$, $\Omega = \underbrace{2\pi f}_{\omega} \cdot \underbrace{\frac{1}{f_s}}_{T} = \omega \cdot T$, $\Omega \in [-\pi, \pi]$

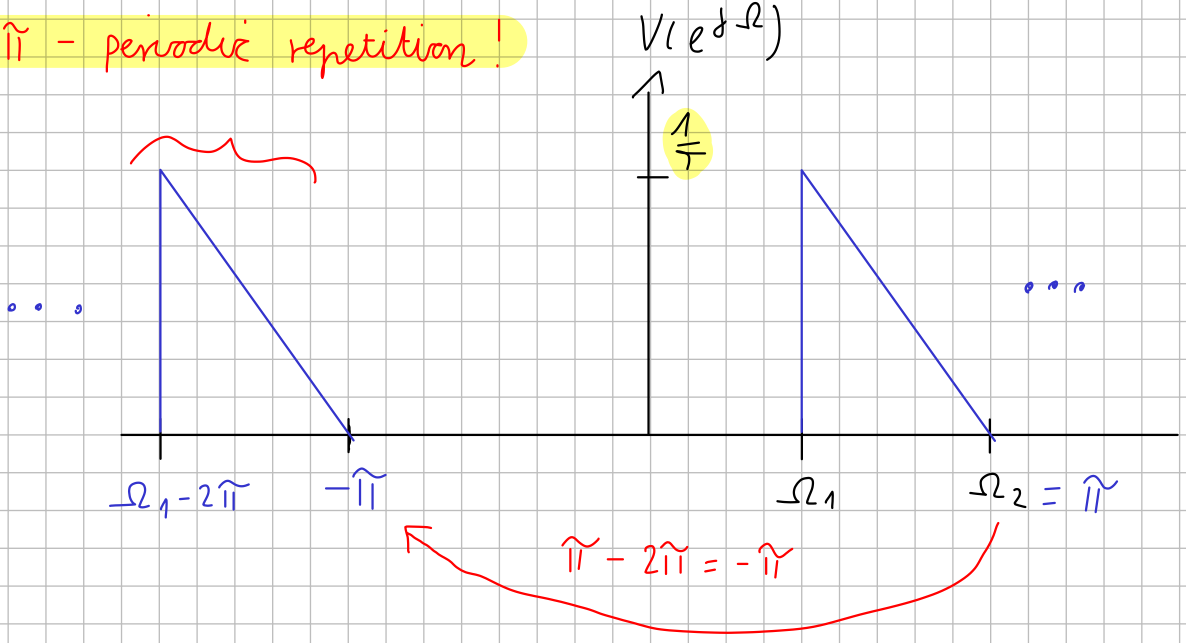
↳ ω normalized to the sampling freq.

$$\Omega_1 = \omega_1 \cdot T = \omega_1 \cdot \frac{\pi}{\omega_2}$$

$$\Omega_2 = \omega_2 \cdot T = \cancel{\omega_2} \cdot \frac{\pi}{\cancel{\omega_2}} = \pi$$

$V(e^{j(\Omega + \lambda \cdot 2\pi)}) = V(e^{j\Omega})$

2π - periodic repetition !



b.) Usually: $\omega_s > 2\,\omega_{max}$

but: Bandpass spectrum with bandwith $\omega_B = \omega_2 - \omega_1$

↳ $\omega_s > 2\omega_B$

e.g. FM-radio: 87,5 MHz to 108 MHz

but: radio channel bandwidth ≈ 15 kHz

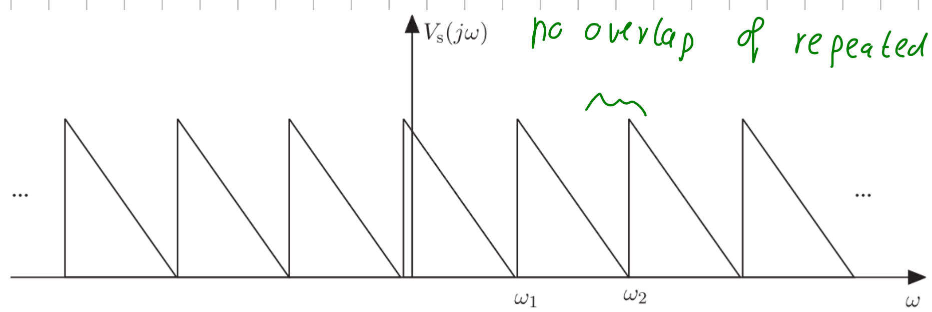
↳ Nyquist freq ≈ 30 kHz

(not ≈ 200 MHz!)

$$\omega_s > 2\,\omega_B$$

$$2\pi f_s > 2 \cdot (\omega_2 - \omega_1)$$

$$f_s > \frac{2(\omega_2 - \omega_1)}{2\pi} \Rightarrow T < \frac{2\pi}{2(\omega_2 - \omega_1)}$$



no overlap of repeated spectra

↳ no Alias

Problem 2 **(overall system for filtering a continuous-time signal in digital domain)**

Figure 2 shows an overall system for filtering a continuous-time signal using a discrete-time filter. The frequency response of the ideal reconstruction filter $H_r(j\omega)$ and the discrete-time filter are shown below.

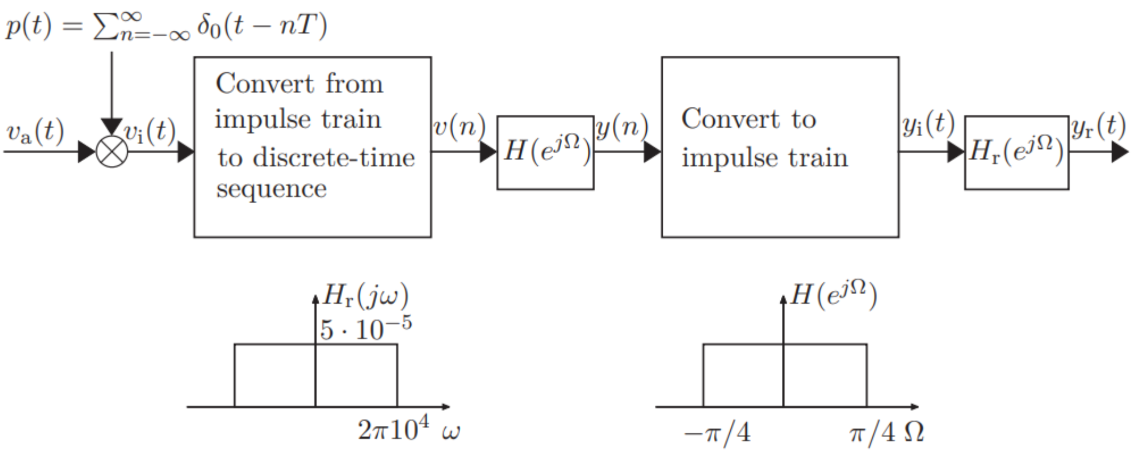


Figure 2: Overall system.

(a) For $V_a(j\omega)$ as shown in figure 3 and $1/T = 20kHz$ sketch $V_i(j\omega)$ and $V(e^{j\Omega})$.

Figure 3: Spectrum of $V_a(j\omega)$ and $H_{\text{eff}}(j\omega)$

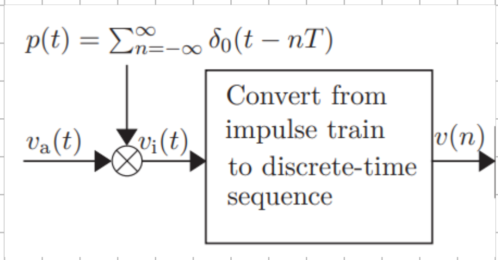
For a certain range of values of T, the overall system, with input $v_a(t)$ and output $y_r(t)$, is equivalent to a continuous-time lowpass filter with frequency response $H_{\text{eff}}(j\omega)$ sketched in figure 3.

(b) Determine the range of values of T for which the information presented above is true, when $V_a(j\omega)$ is bandlimited to $|\omega| \leq 2\pi \cdot 10^4$ as shown in figure 3.

(c) For the range of values determined in (b), sketch ω_c as a function of $1/T$.

Note: This is one way of implementing a variable-cutoff continuous-time filter using fixed continuous-time and discrete-time filters and a variable sampling rate.

2a.)



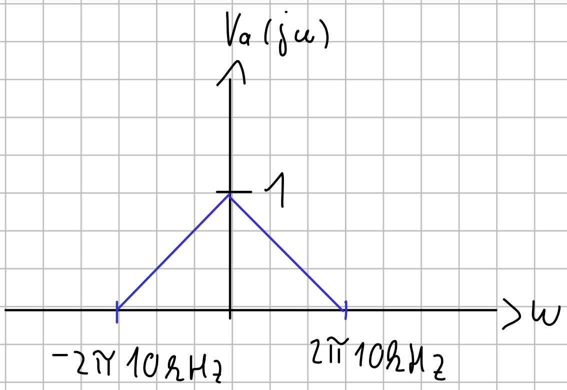
} Time domain mult with dirac impulse train ○—● convolution in FD

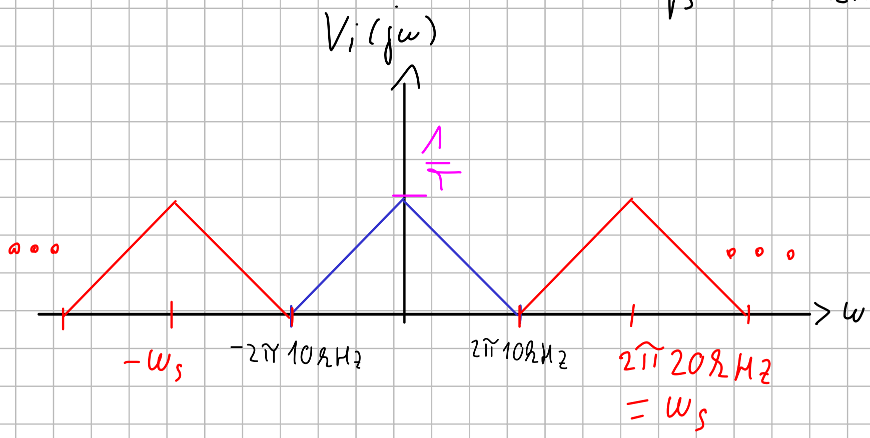
↳ $V_i(j\omega) = \frac{1}{T} \sum_{-\infty = n}^{\infty} V_a(j(\omega - n \cdot \omega_s))$

scaling

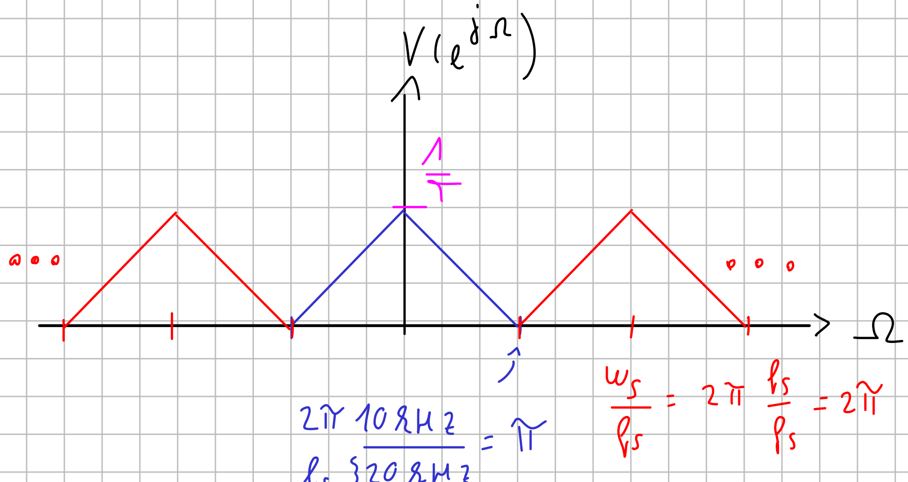
periodic repetition at ω_s multiples

↳ $\omega_s = 2\pi f_s = 2\pi\, 20\,kHz$





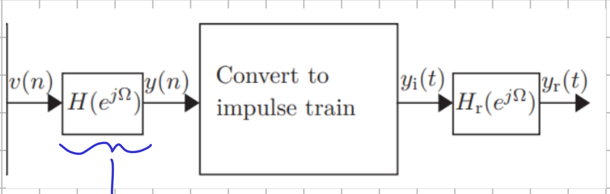
Now: discretize signal: $\Omega = \omega T = \frac{\omega}{f_s}$

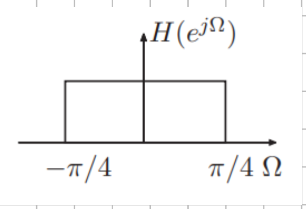


b.) Which sampling rate range to ensure LP-behavior

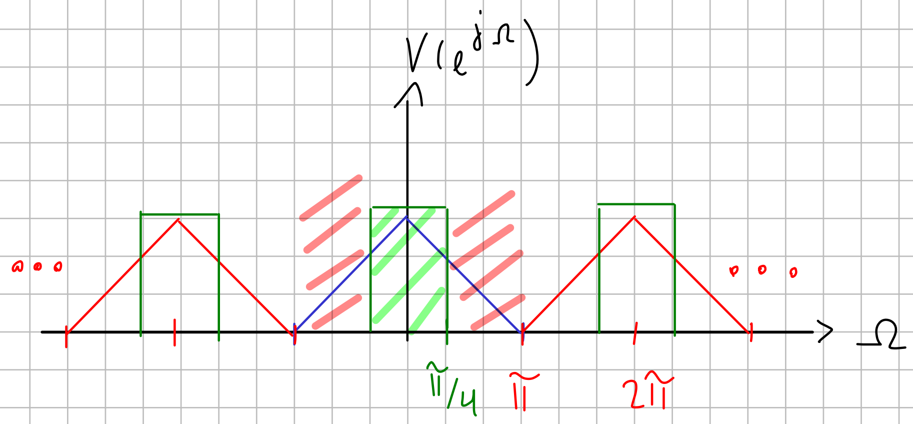
No aliasing in LP-range!

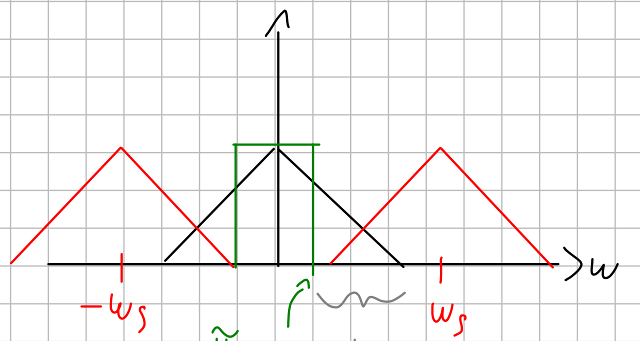
↳ aliasing outside of LP doesn't matter



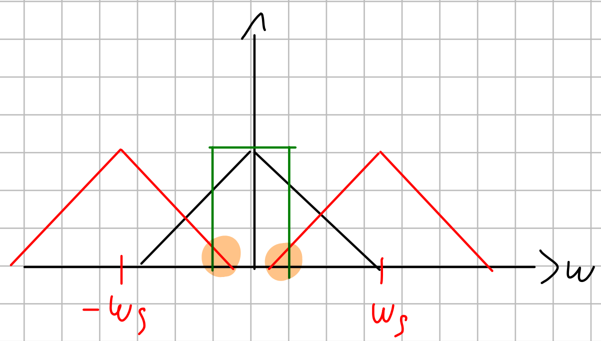


LP with $\pi/4$ cut-off:





$\frac{\pi}{4} f_s$ aliasing outside of LP range ✓



aliasing in LP range ↯

$$\hookrightarrow \frac{\pi}{4} \cdot f_s \leq \underbrace{2\pi f_s}_{w_s} - 2\pi\, 10\,kHz$$

alternatively:

$$\frac{\pi}{4} \leq 2\pi - 2\pi\, 10\,kHz \cdot T$$

...

$$T \leq 8{,}75 \cdot 10^{-5}\, s$$

$$2\pi\, 10\,kHz \leq f_s \left(2\pi - \frac{\pi}{4}\right)$$

$$f_s \geq \frac{2\pi\, 10\,kHz}{2\pi - \frac{\pi}{4}} \approx 11{,}42\,kHz$$

$$\Rightarrow T \leq 8{,}75 \cdot 10^{-5}\ s$$

! If signal completely in LP range, LP unnecessary

↳ if $|\Omega| \leq \frac{\pi}{4}$ $\Rightarrow$ $\underbrace{2\pi \, 10kHz \cdot T}_{\omega \cdot T} \geq \frac{\pi}{4}$

↳ $T \geq 1{,}25 \cdot 10^{-5}\,s \rightsquigarrow f \leq 80kHz$

c.) ω_c as a function of $\frac{1}{T}$

$\Omega_c = \omega_c \cdot T$

$\omega_c = \frac{\Omega_c}{T} = \Omega_c \cdot f_s$

↳ $\frac{\pi}{4} \cdot \underbrace{f_s}_{1/T} = \omega_c$

↳ $\omega_c = \underbrace{\frac{\pi}{4}}_{\text{slope}} \cdot \frac{1}{T}$