ADSP Ex. 3

DFT:

$$V_M(\mu) = \mathrm{DFT}_M\{v(n)\} = \sum_{n=0}^{M-1} v(n)\, e^{-j\mu\frac{2\pi}{M}n}$$

Problem 4 (DFT and convolution)

$M_2 = 2$

Let $h(n)$ be the sequence $\{1,1,0,0,0,0,0,0\}$ and $y(n) = \{1,1,1,1,0,0,0,0\}$.

(a) Calculate the DFT of length 8 for both sequences. $\rightsquigarrow M = 8$

a.) $$H_{M=8}(\mu) = \sum_{n=0}^{8-1} h(n)\cdot \underbrace{e^{-j\mu\frac{2\pi}{M}n}}_{W_M^{\mu n}}$$

$$= 1\cdot \underbrace{e^{-j\mu\frac{2\pi}{8}0}}_{e^0 = 1} + 1\cdot e^{-j\mu\frac{2\pi}{8}\cdot 1} = 1 + e^{-j\mu\frac{2\pi}{8}}$$

$$= \underbrace{1}_{W_8^0} + W_8^{\mu} = 1 + e^{-j\mu\frac{\pi}{4}}$$

$$Y_8(\mu) = \sum_{n=0}^{8-1} y(n)\cdot e^{-j\mu\frac{2\pi}{M}n}$$

$$= e^0 + e^{-j\mu\frac{2\pi}{8}} + e^{-j\mu\frac{2\pi}{8}\cdot 2} + e^{-j\mu\frac{2\pi}{8}\cdot 3}$$

$$= 1 + W_8^{\mu} + W_8^{2\mu} + W_8^{3\mu} = 1 + e^{-j\mu\frac{\pi}{4}} + e^{-j\mu\frac{\pi}{2}} + e^{-j\mu\frac{3\pi}{4}}$$

$$= \left(1 + e^{-j\frac{\pi}{4}\mu}\right)\left(1 + e^{-j\frac{\pi}{2}\mu}\right)$$

b.) (b) Determine with help of the DFT a sequence $v(n)$ such that $y(n) = h(n) \circledast v(n)$.

$\circledast$: Circular convolution

$$y(n) = v(n) \circledast h(n) = \sum_{k=0}^{7} v(k)\, h(k-n)$$

$$\circ\!\!-\!\!\bullet$$

$$Y(\mu) = V(\mu)\cdot H(\mu) \rightsquigarrow V(\mu) = \frac{Y(\mu)}{H(\mu)}$$

$$= \frac{1 + W_8^{\mu} + W_8^{2\mu} + W_8^{3\mu}}{1 + W_8^{\mu}}$$

$$= \frac{\left(1 + e^{-j\frac{\pi}{4}\mu}\right)\left(1 + e^{-j\frac{\pi}{2}\mu}\right)}{1 + e^{-j\mu\frac{\pi}{4}}} = 1 + e^{-j\frac{\pi}{2}\mu} = V(\mu)$$

$$\bullet\!\!-\!\!\circ$$

$$v(n) = \frac{1}{M}\sum_{\mu=0}^{M-1} V_M(\mu)\, e^{j\mu\frac{2\pi}{M}\cdot n}$$

$$V(n) = \frac{1}{8} \cdot \sum_{\mu=0}^{7} \left(1 + e^{-j\frac{\pi}{2}\mu}\right) \cdot e^{j\mu\frac{\pi}{4}n}$$

$$= \frac{1}{8} \cdot \Big(\underbrace{(1+1)}_{\mu=0} + \underbrace{\left(e^{j\frac{\pi}{4}n} + e^{j\left(\frac{\pi}{4}n - \frac{\pi}{2}\right)}\right)}_{\mu=1}$$

$$+ \underbrace{\left(e^{j\frac{\pi}{2}n} + e^{j2\cdot\left(\frac{\pi}{4}n - \frac{\pi}{2}\right)}\right)}_{\mu=2} + \underbrace{\left(e^{j\frac{3\pi}{4}n} + e^{j3\left(\frac{\pi}{4}n - \frac{\pi}{2}\right)}\right)}_{\mu=3} \ldots$$

↳ Insert $n = \{0, \ldots, 7\}$

↳ $V(n) = \{\underbrace{1, 0, 1}_{M_1 = 3}, 0, 0, 0, 0, 0\}$

c.) (c) Let $z(n)$ be the result of the linear convolution of $h(n)$ and $v(n)$: $z(n) = h(n) * v(n)$
Is $z(n) = y(n)$?

Signal lengths (without zeros)

$V(n)$: $M_1 = 3$, $h(n)$: $M_2 = 2$

Linear $*$ Circular $\circledast$

Length: $\underbrace{M_1 + M_2 - 1}_{M_L}$ $\underbrace{\max(M_1, M_2)}_{M_C}$

Equivalent if $M_L \leq M_C$

Here: $M_1 + M_2 - 1 = 3 + 2 - 1 = 4 \leq 8$ ✓

↳ lin. conv. result $z(n) = y(n)$, circ. conv. result

↳ Efficient realization of filtering operation in freq. domain if signals are zero padded to length M_L of lin. conv.

↳ DFT, FFT

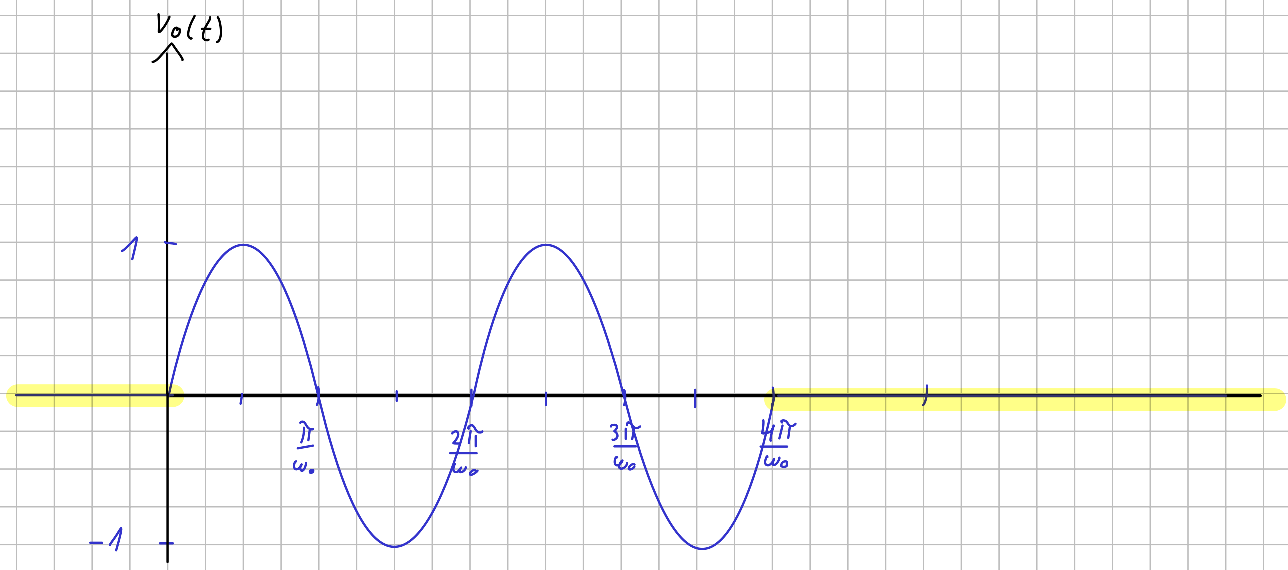
Problem 5 (DFT)

The time-limited signal

$$v_0(t) = \begin{cases} \sin(\omega_0 t) & for \quad 0 \le t < 4\pi/\omega_0 \\ 0 & otherwise \end{cases}$$

is sampled with $t_n = nT_\mathrm{A} = n\frac{\pi}{4\omega_0}$ to produce the time-limited sequence $v(n)$.

(a) Sketch $v_0(t)$.



(b) Determine $v(n)$.

$v(n) \hat{=} v_0(t)$ sampled at $t_n = n \cdot T_A$, $T_A = \frac{\pi}{4\omega_0}$, $f_A = \frac{4\omega_0}{\pi}$

$$v(n) = v_0(t)\Big|_{t=n\cdot T_A} = v_0(n \cdot T_A) = \begin{cases} \sin\left(\omega_0 \cdot n \cdot \frac{\pi}{4\omega_0}\right) = \sin\left(n \cdot \frac{\pi}{4}\right), & 0 \le n < \underset{\sim}{16} \\ 0, & else \end{cases}$$

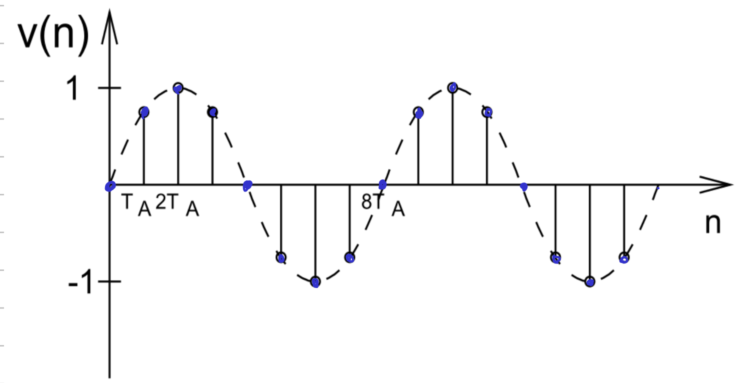
$\frac{4\pi}{\omega_0} \hat{=} n \cdot \frac{\pi}{4\omega_0}$

$\Rightarrow n = 16$

$\omega_0 = 2\pi f_0 \Rightarrow f_0 = \frac{\omega_0}{2\pi}$, $f_A = \frac{4\omega_0}{\pi}$

$$\frac{f_A}{f_0} = \frac{4\cancel{\omega_0}}{\cancel{\pi}} \cdot \frac{2\cancel{\pi}}{\cancel{\omega_0}} = 8 \rightsquigarrow f_A = 8 \cdot f_0$$

$(f_A \ge 2 f_0 \ !)$



(c) Determine the DFT of $v(n)$

$$v(n) = \left\{ 0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, \dots \right\}$$

$$V(\mu) = \mathrm{DFT}_{M=16}\{v(n)\} = \sum_{n=0}^{15} v(n) W_{16}^{n\mu}$$

$$W_M = e^{-j\frac{2\pi}{M}}$$

$$= 0 \cdot W_{16}^{0\mu} + \boxed{\frac{1}{\sqrt{2}} \cdot W_{16}^{1\mu} + 1 \cdot W_{16}^{2\mu} + \frac{1}{\sqrt{2}} W_{16}^{3\mu}}$$

$$- 0\, W_{16}^{4\mu} - \frac{1}{\sqrt{2}} W_{16}^{5\mu} - 1\, W_{16}^{6\mu} - \frac{1}{\sqrt{2}} W_{16}^{7\mu} \qquad \cdot(-W_{16}^{4\mu})$$

$$+ 0\, W_{16}^{8\mu} + \frac{1}{\sqrt{2}} W_{16}^{9\mu} + 1\, W_{16}^{10\mu} + \frac{1}{\sqrt{2}} W_{16}^{11\mu} \qquad \cdot W_{16}^{8\mu}$$

$$- 0\, W_{16}^{12\mu} - \frac{1}{\sqrt{2}} W_{16}^{13\mu} - 1\, W_{16}^{14\mu} - \frac{1}{\sqrt{2}} W_{16}^{15\mu} \qquad \cdot(-W_{16}^{12\mu})$$

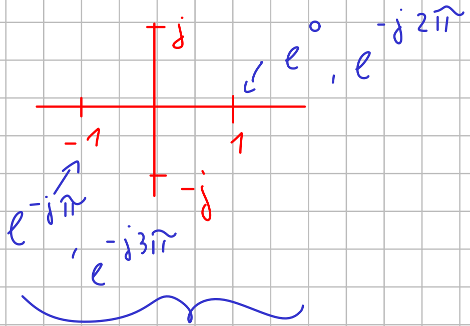
Factorize: $\underbrace{\left(1 - W_{16}^{4\mu} + W_{16}^{8\mu} - W_{16}^{12\mu}\right)} \cdot \boxed{\left(\frac{1}{\sqrt{2}} W_{16}^{\mu} + W_{16}^{2\mu} + \frac{1}{\sqrt{2}} W_{16}^{3\mu}\right.}$

Factorize:

$$\left(1 + W_{16}^{8\mu}\right)\left(1 - W_{16}^{4\mu}\right)$$

$$= \underbrace{\left(1 + W_{16}^{8\mu}\right)}_{\mathrm{I}} \underbrace{\left(1 - W_{16}^{4\mu}\right)}_{\mathrm{II}} \cdot \underbrace{\left(\frac{1}{\sqrt{2}} W_{16}^{\mu} + W_{16}^{2\mu} + \frac{1}{\sqrt{2}} W_{16}^{3\mu}\right)}_{\mathrm{III}}$$

$$W_{16}^{8\mu} = e^{-j\frac{2\pi}{16} \cdot 8\mu} = e^{-j\pi \cdot \mu}$$



$$1 + W_{16}^{8\mu} = \begin{cases} 2, & \mu \text{ even} \\ 0, & \mu \text{ odd} \end{cases}$$

II.) $1 - W_{16}^{4\mu} = \begin{cases} 0, & \mu = 0, 4, 8, 12 \\ 1-(-j)^{\mu}, & \text{else} \end{cases}$

From I.) & II.): $V(\mu)$ zero if odd & $\in \{0, 4, 8, 12\}$

↳ not zero for $\underbrace{\mu = \{2, 6, 10, 14\}}_{\text{Only compute III for these}}$

And: Due to symmetry of DFT: $V(\mu) = V^*(M-\mu)$

Matlab: fftshift(fft(...))

↳ only 2 & 6

III: $\mu = 2: \frac{1}{\sqrt{2}} W_{16}^{2} + W_{16}^{4} + \frac{1}{\sqrt{2}} W_{16}^{6} = -2j$

$\mu = 6: \frac{1}{\sqrt{2}} W_{16}^{6} + W_{16}^{12} + \frac{1}{\sqrt{2}} W_{16}^{18} = 0$

↳ $V(2) = 2 \cdot 2 \cdot (-2j) = -8j \rightsquigarrow V(14) = V^*(16-14)$

$= V^*(2) = +8j$

$V(6) = 2 \cdot 2 \cdot 0 = 0$

↳ $V(10) = V^*(16-10) = V^*(6) = 0$

$$V(\mu) = \begin{cases} -8j & \mu = 2 \\ 8j & \mu = 14 \\ 0 & \text{otherwise} \end{cases}$$

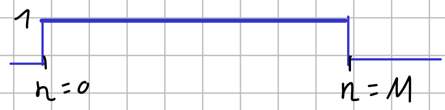
(d) Determine the Fourier Transform $V(e^{j\Omega})$ of $v(n)$

periodic, continuous

non-periodic, discrete

$\hookrightarrow \sin(\frac{\pi}{4} \cdot n)$ only in $n \in \{0, \dots, \underbrace{M-1}_{15}\}$

$$v(n) = \sin(\frac{\pi}{4} \cdot n) \cdot \overbrace{(\gamma_{-1}(n) - \gamma_{-1}(n-M))}^{f_M(n)}$$



$$\stackrel{!}{=} \begin{cases} \sin(\dots), & n = 0 \dots 15 \\ 0, & \text{else} \end{cases}$$

$$DTFT\{f_M(n)\} = \sum_{n=0}^{M-1} 1 \cdot e^{-j\Omega n}$$
$$= \frac{1 - e^{-j\Omega M}}{1 - e^{-j\Omega}} = \frac{e^{-j\frac{\Omega M}{2}}(e^{j\frac{M\Omega}{2}} - e^{-j\frac{M\Omega}{2}})}{e^{j\frac{\Omega}{2}}(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}})}$$
$$= e^{-j\Omega(\frac{M-1}{2})} \cdot \frac{sin(\frac{\Omega M}{2})}{sin(\frac{\Omega}{2})}$$

$$V(e^{j\Omega}) = \frac{1}{2\pi} \underbrace{\{ j\pi [\delta_0(\Omega + \frac{\pi}{4}) - \delta_0(\Omega - \frac{\pi}{4})]\}}_{DTFT\{\sin(\frac{\pi}{4} \cdot n)\}} * e^{-j\Omega(\frac{M-1}{2})} \cdot \frac{\sin(\frac{\Omega M}{2})}{\sin(\frac{\Omega}{2})}$$

$$= \frac{j}{2}\left\{ e^{-j\frac{(M-1)}{2}(\Omega + \frac{\pi}{4})} \cdot \frac{\sin[(\Omega + \frac{\pi}{4})\frac{M}{2}]}{\sin[\frac{\Omega + \frac{\pi}{4}}{2}]} - e^{-j\frac{(M-1)}{2}(\Omega - \frac{\pi}{4})} \cdot \frac{\sin[(\Omega - \frac{\pi}{4})\frac{M}{2}]}{\sin[\frac{\Omega - \frac{\pi}{4}}{2}]} \right\}$$

(e) Explain the connection between the DFT$\{v(n)\}$ and $V(e^{j\Omega})$

$$V(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} v(n) \cdot f_M(n) . e^{-j\Omega n} = \sum_{n=0}^{15} v(n) \cdot e^{-j\Omega n}$$

$$DFT\{v(n)\} = V(\mu) \stackrel{\wedge}{=} V(e^{j\Omega})\Big|_{\Omega = \frac{2\pi}{M} \cdot \mu}, \quad \mu = 0, \dots, 15$$

DFT samples the DTFT at $\Omega = \dots$