ADSP - Exercise 4: Blocks and Windows

Learning objectives

After this exercise, you should be able to

- explain how Overlap-Add and Overlap-Save produce the linear convolution of a long signal from DFT blocks,
- derive the DFT-length condition and the valid Overlap-Save outputs,
- interpret a block as a windowed section of the signal and explain leakage as a consequence of the finite observation,
- derive the spectrum of a windowed sequence as a convolution in frequency, and
- relate main-lobe width and sidelobe level to the resolution of the rectangular and the Hann window.

Guiding question: Cutting a signal into blocks does not disturb linear filtering, but it does change the spectrum of each block. Why?

Notation (lecture notes, Part 4, slides 11-35): $v(n)$ is the input, $h(n)$ is a causal FIR impulse response of length M_2, M is the DFT length, and M_1 is the number of new input samples per block. The block index is ν, the frequency index is μ, and $y_\nu(r)$ is a local IDFT output with local time r; n is global time. From Section 4 on, L is the length of an observation window $w(n)$, which is zero outside $0 \le n \le L-1$; the windowed sequence is $\hat{v}(n) = v(n)\,w(n)$ with spectrum $\hat{V}(e^{j\Omega})$, and $W(e^{j\Omega})$ is the spectrum of the window. Finite recordings start at $n = 0$ and are zero before and after their stated range. The notebook `Uebung_04_Demo.ipynb` reproduces all examples of this sheet.

1. Reactivation: what must fit into one period?

Consider the block $\{1, 2, 3, 4\}$ and $h(n) = \{1, 1, 1\}$, i.e. $M_1 = 4$ and $M_2 = 3$. State in one sentence what multiplying two M-point DFTs and applying an IDFT computes, and which condition on M prevents folding. For $M = 4$, which local output indices fold, and to which positions?

2. Overlap-Add

Consider the finite recording

$$v(n) = \{1, 2, 3, 4, 2, 0, -1, 1\}.$$

Separate it into $v_0 = \{1, 2, 3, 4\}$ and $v_1 = \{2, 0, -1, 1\}$, with local indices starting at zero. Their linear convolutions with h are supplied:

$$y_0 = \{1, 3, 6, 9, 7, 4\}, \qquad y_1 = \{2, 2, 1, 0, 0, 1\}.$$

1. Use linearity and time shifting to complete $y(n) = \sum_\nu y_\nu(\ldots)$. Why is the shift determined by M_1 and not by M?
2. Place the two results at global indices 0-5 and 4-9. Determine $y(4)$ and $y(5)$. Explain why the two local results cannot simply be concatenated.
3. Derive a condition on M so that each local result can be computed by DFT multiplication without circular aliasing. Why can the later global addition not repair a block computed with $M = 4$?

3. Overlap-Save

Keep $M_2 = 3$ and choose $M = 6$. Use the smallest sufficient input overlap, $M_2 - 1 = 2$. The first blocks are

$$v_0 = \{0, 0, 1, 2, 3, 4\}, \qquad v_1 = \{3, 4, 2, 0, -1, 1\}.$$

For the second block, z_1 denotes its local linear reference and y_1 its six-point circular result:

$$z_1 = \{3, 7, 9, 6, 1, 0, 0, 1\}, \qquad y_1 = \{3, 8, 9, 6, 1, 0\}.$$

1. Fold local tail indices 6 and 7 modulo 6. Mark the potentially invalid prefix and the valid suffix. Does every discarded value necessarily differ numerically from the reference?
2. For local output r, the filter uses input indices $r-\kappa$, $0 \le \kappa < M_2$. Derive the range of r for which all required input samples are inside the block. How many outputs can be kept per block?
3. Map the retained values of y_1 to global output indices. Which input samples must be saved for the next block?
4. For $M = 8$ instead of 6, state the overlap, the number of new samples per block and the number of retained outputs. Why does a larger block alone not establish a faster implementation?

4. A block is a windowed section

In Overlap-Add, block ν is $v_\nu(r) = v(r + \nu M_1)\, w(r)$ with the rectangular window $w(r) = 1$ for $0 \le r < M_1$ and zero elsewhere.

1. Show that $\sum_\nu w(n - \nu M_1) = 1$ for all $n \ge 0$. Explain why the assembled Overlap-Add output is exact although every block is cut off abruptly.
2. Instead of filtering, we now compute the DFT of a single block to analyse the signal. What does the window do to the spectrum of $v(n)$? Sections 5-7 answer this question.

5. Leakage: why a single cosine is spread (slides 18-26)

The lecture uses $M = L = 50$, a rectangular window and

$$v_0(n) = \cos\left(1.0\,\frac{2\pi}{L}\,n\right), \qquad v_1(n) = \cos\left(1.2\,\frac{2\pi}{L}\,n\right).$$

1. Sketch the periodic continuation of both records with period M. Where does a discontinuity occur? Predict for which sequence only two DFT values are nonzero.
2. Explain why $\hat{V}_0(e^{j\Omega_\mu}) = 0$ for all $\mu \neq 1,\, M-1$, although $\hat{V}_0(e^{j\Omega})$ is not zero between these samples.
3. What does zero-padding to $M = 200$ change for v_0 and for v_1? Does it remove leakage?

6. The windowed spectrum is a convolution (slides 19-21)

1. Insert the inverse DTFT $v(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(e^{j\theta}) e^{j\theta n}\, d\theta$ into $\hat{V}(e^{j\Omega}) = \sum_n v(n) w(n) e^{-j\Omega n}$ and show

$$\hat{V}(e^{j\Omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(e^{j\theta})\, W(e^{j(\Omega-\theta)})\, d\theta.$$

2. For $v(n) = \cos(\Omega_0 n)$ the lecture gives $V(e^{j\Omega}) = \pi\big[\delta_0(\Omega-\Omega_0) + \delta_0(\Omega+\Omega_0)\big]$. Derive $\hat{V}(e^{j\Omega})$.
3. Evaluate $W(e^{j\Omega})$ for the rectangular window as a geometric sum. Where is its first zero, and what happens to the main lobe when L is doubled?

7. Two close frequencies: resolution and window choice (slides 27-35)

The lecture example is $v(n) = \cos(\Omega_0 n) + \cos(\Omega_1 n) + \cos(\Omega_2 n)$, observed for $n = 0, \dots, L-1$, with $\Omega_0 = 0.2\pi$, $\Omega_1 = 0.22\pi$, $\Omega_2 = 0.6\pi$. The Hann window is

$$w_{\text{Hann}}(n) = \tfrac{1}{2}\left[1 - \cos\left(\frac{2\pi}{L-1}\,n\right)\right], \qquad 0 \le n \le L-1,$$

and zero elsewhere (slide 33).

1. Apply the criterion of slide 29, $2\pi/L$ compared with $|\Omega_1 - \Omega_0|$: for which $L \in \{25, 50, 100, 150, 200\}$ are the main lobes of the rectangular window separated?
2. At $L = 100$, both windows show two maxima (slides 32 and 35), although $2\pi/L = |\Omega_1 - \Omega_0|$. Explain with the complex sum of the two shifted window spectra.
3. The first zero of the Hann spectrum lies at about $\pm 4\pi/L$. Estimate the observation length the Hann window needs for this tone spacing. How do the required lengths change for $\Omega_1 = 0.26\pi$?

4. Now $v(n) = \cos(0.2\pi n) + 0.01\cos(0.33\pi n)$ with $L = 100$. Which window shows the weak component? Explain with main lobe and sidelobes.
5. Slide 27 asks why a longer sequence is not always the better choice. Give two reasons.

8. Peer oral examination

Work in pairs using examination cards A and B.

- 3 min: Student A examines student B using card A.
- 3 min: Change roles; student B examines student A using card B.
- 2 min: Give feedback using the checklist.

9. Take-away

Complete verbally: **Overlap-Add prevents local aliasing by ... and combines ... . Overlap-Save includes ... and retains ... . Cutting a block multiplies the signal by ..., so its spectrum is ... . The window length determines ..., the window shape determines ... .**