ADSP - Ex. 4

Let $v(n)$ be a time-discrete signal

$$v(n) = [v(0), v(1), v(2), v(3), v(4), v(5), v(6), v(7)].$$

(a) Separate the signal $v(n)$ into even and odd time-indices $v_1(n)$ and $v_2(n)$ respectively and find the DFT expression for each separated sequence.

$$V_1(n) := V_{even}(n) = [\underbrace{V(0)}_{n=0}, \underbrace{V(2)}_{n=1}, V(4), V(6)] \quad \ldots$$

$$V_2(n) := V_{odd}(n) = [V(1), V(3), V(5), V(7)]$$

$$V_M(\mu) = \mathrm{DFT}_M\{v(n)\} = \sum_{n=0}^{M-1} v(n)\, e^{-j\mu\frac{2\pi}{M}n}$$

$$\underbrace{V_{even, M=4}(\mu) = \mathrm{DFT}\{V_{even}(n)\} = \sum_{n=0}^{4-1} V_{even}(n)\, \underbrace{e^{-j\mu\frac{2\pi}{4}n}}_{W_4^{\mu n}}}$$

$$\hat{=} \sum_{n=0}^{3} V(2\cdot n) \cdot W_8^{\mu(2\cdot n)}$$

$$V_{odd, M=4}(\mu) = \mathrm{DFT}\{V_{odd}(n)\} = \sum_{n=0}^{4-1} V_{odd}(n)\, \underbrace{e^{-j\mu\frac{2\pi}{4}n}}_{W_4^{\mu n}}$$

$$\hat{=} \sum_{n=0}^{3} V(2\cdot n+1) \cdot W_8^{\mu\cdot(2\cdot n+1)}$$

(b) Now compute the DFT of $v(n)$ using the above expressions.

$$V_8(\mu) = \underbrace{\sum_{n=0}^{8-1} V(n) \cdot W_8^{\mu n}}_{\text{Split into:}} = \underbrace{\sum_{n=0}^{3} V(2\cdot n) \cdot W_8^{\mu(2\cdot n)}}_{\text{even}} + \underbrace{\sum_{n=0}^{3} V(2\cdot n+1) \cdot W_8^{\mu\cdot(2\cdot n+1)}}_{\text{odd}}$$

$$W_8^{\mu(2\cdot n)} = e^{-j\mu\frac{2\pi}{8}\cdot 2n} = e^{-j\mu\frac{2\pi}{4}n} = W_4^{\mu n}$$

} Twiddle factor of M=4 DFT!

$$W_8^{\mu(2n+1)} = e^{-j\mu\frac{2\pi}{8}(2n+1)} = e^{-j\mu\frac{2\pi}{4}n} \cdot e^{-j\mu\frac{2\pi}{8}} = W_4^{\mu n} \cdot W_8^{\mu}$$

$$V_8(\mu) = \underbrace{\sum_{n=0}^{3} V_{even}(n) \cdot W_4^{\mu\cdot n}}_{M=4\ \mathrm{DFT}} + \underbrace{W_8^{\mu} \cdot \sum_{n=0}^{3} V_{odd}(n) \cdot W_4^{\mu\cdot n}}_{M=4\ \mathrm{DFT}}$$

↳ Split 8-point DFT into 2x 4-point DFT!

(c) Sketch the signal flow diagrams when DFT is directly applied to $v(n)$ and as shown in part (b). Show the reduction in complexity by computing the number of complex multiplications for each method.

$$V_8(\mu) = \underbrace{\sum_{n=0}^{8-1}}_{M\text{-times}} \underbrace{v(n)\cdot W_8^{\mu n}}_{1 \text{ complex mult}}$$

M-complex mult for

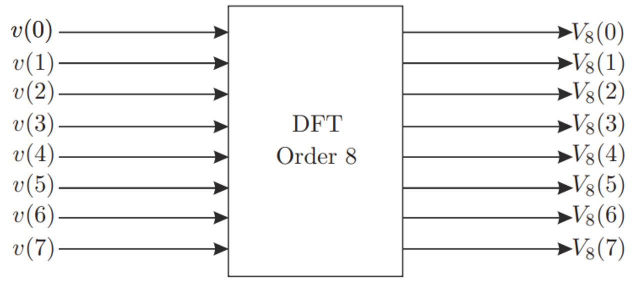
$V_M(0)$
$V_M(1)$
$\vdots$
$V_M(M-1)$

M-times

$$\underbrace{M\cdot M = M^2}_{O(n^2)}$$

quadratic complexity

$8^2 = 64$ cplx. mult



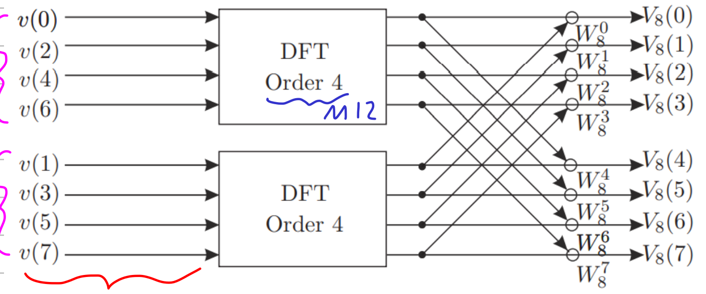
$$V_8(\mu) = \underbrace{\sum_{n=0}^{3} V_{even}(n)\cdot W_4^{\mu\cdot n}}_{\frac{M}{2}} + \underbrace{W_8^{\mu}}_{M\text{-times}} \cdot \underbrace{\sum_{n=0}^{3} V_{odd}(n)\cdot W_4^{\mu\cdot n}}_{\frac{M}{2}}$$

$\frac{M}{2}$ times

↳ $2\cdot\left(\frac{M}{2}\right)^2 + M \rightsquigarrow 2\cdot\left(\frac{8}{2}\right)^2 + 8 = 40$

Saved 37,5% of operations!

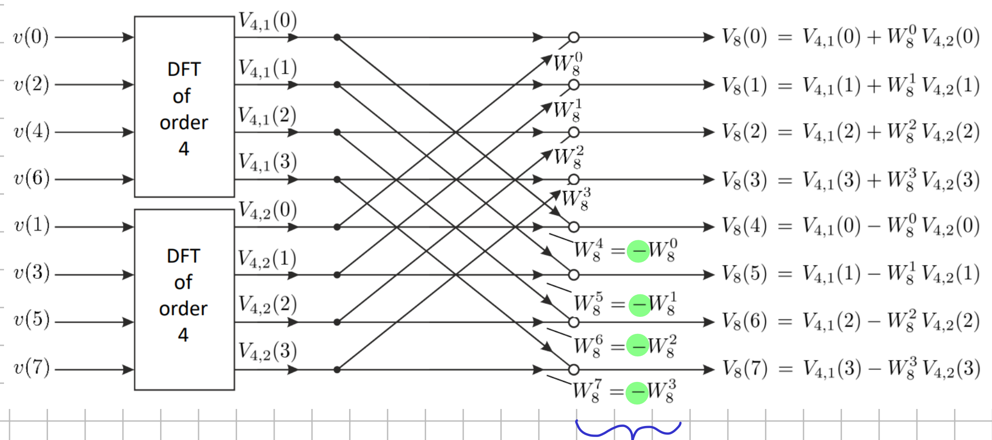
2x



$\frac{M}{2}$

$2\cdot\left(\frac{M}{2}\right)^2$

M



Twiddle-factor symmetry: Only Sign changes

(d) Can the complexity be reduced further? If yes then find the final expression.

Yes, by applying the same principle again

$$V_1(n) := V_{even}(n) = [v(0),\ v(2),\ v(4),\ v(6)] \begin{cases} \xrightarrow{\text{even}} V_{1,1}(n) = [v(0),\ v(4)] \\ \xrightarrow{\text{odd}} V_{1,2}(n) = [v(2),\ v(6)] \end{cases}$$

$n = \quad 0 \quad 1 \quad 2 \quad 3$

$$V_2(n) := V_{odd}(n) = [v(1),\ v(3),\ v(5),\ v(7)] \begin{cases} \xrightarrow{\text{even}} V_{2,1}(n) = [v(1),\ v(5)] \\ \xrightarrow{\text{odd}} V_{2,2}(n) = [v(3),\ v(7)] \end{cases}$$

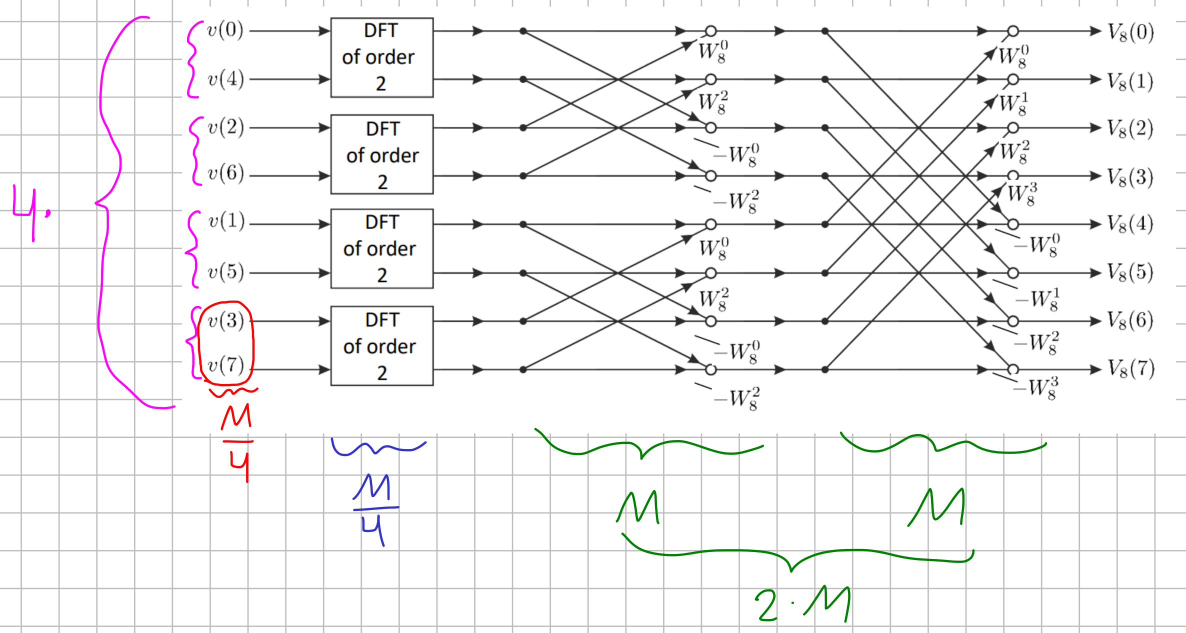
$$\hookrightarrow V_8(\mu) = \sum_{n=0}^{1} V_1(2n) \cdot W_4^{\mu(2n)} + \sum_{n=0}^{1} V_1(2n+1) \cdot W_4^{\mu(2n+1)}$$

$$+ \sum_{n=0}^{1} V_2(2n+1) \cdot W_4^{\mu(2n)} + \sum_{n=0}^{1} V_2(2n+1) \cdot W_4^{\mu(2n+1)}$$

$$= \underbrace{\sum_{n=0}^{1} V_{1,1}(n) \cdot W_2^{n\mu}}_{V_{1,1}(\mu)} + W_4^{\mu} \underbrace{\sum_{n=0}^{1} V_{1,2}(n) \cdot W_2^{n\mu}}_{V_{1,2}(\mu)}$$

$$+ \underbrace{\sum_{n=0}^{1} V_{2,1}(n) \cdot W_2^{n\mu}}_{\text{DFT} \rightsquigarrow V_{2,1}(\mu)} + W_4^{\mu} \underbrace{\sum_{n=0}^{1} V_{2,2}(n)\, W_2^{n\mu}}_{V_{2,2}(\mu)}$$

$$= V_{1,1}(\mu) + W_4^{\mu} \cdot V_{1,2}(\mu) + V_{2,1}(\mu) + W_4^{\mu} \cdot V_{2,2}(\mu)$$



$$4 \cdot \left(\frac{M}{4}\right)^2 + 2M$$

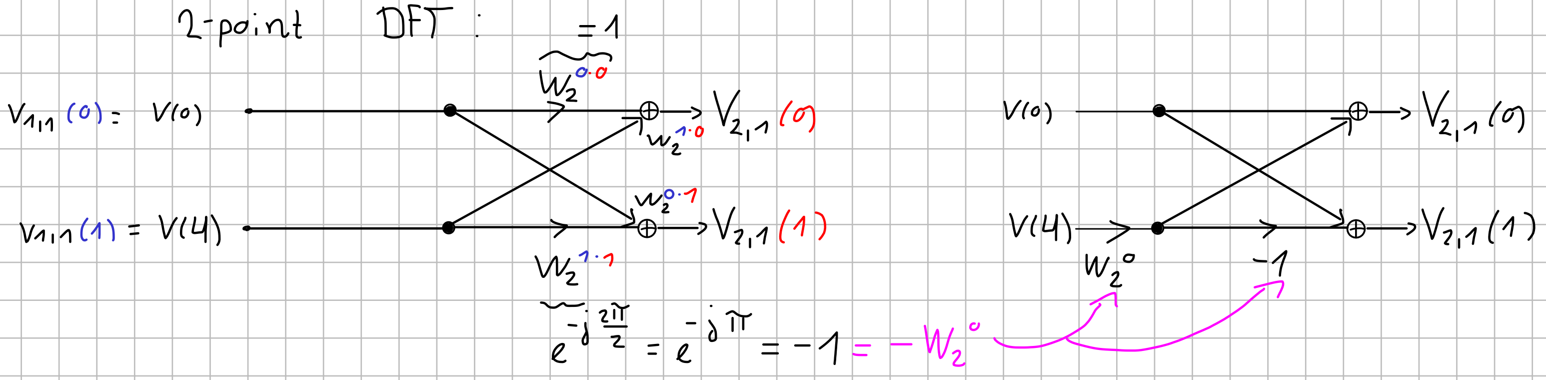
$$4 \cdot \frac{M^2}{16} + 2M$$

$$= \underbrace{\frac{M^2}{4} + 2M}$$

32 operations for $M = 8$

Further complexity reduction:

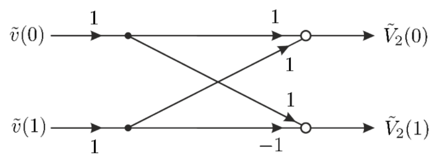
2-point DFT:



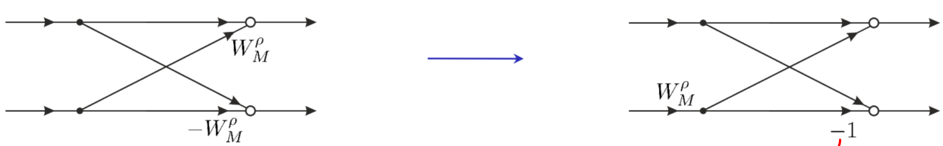
$$\tilde{V}_2(0) = \tilde{v}(0) + W_2^0\,\tilde{v}(1) = \tilde{v}(0) + \tilde{v}(1),$$

$$\tilde{V}_2(1) = \tilde{v}(0) + W_2^1\,\tilde{v}(1) = \tilde{v}(0) - \tilde{v}(1).$$

As we can see, also over here we need the same ***basic scheme*** that we have used also in the previous decompositions:



This basic scheme is called „***butterfly***“ of a radix-2 FFT. The abbreviation ***FFT*** stands for ***Fast Fourier Transform***.



Twiddle factor symmetry

↳ $-1 \hat{=}$ sign-bit inversion

only 1 complex mult

50% complexity reduction

$$\boxed{\text{DFT}: M^2 \text{ cplx. mult} \longrightarrow \text{FFT}: M \cdot \log_2(M) \text{ cplx. mult}}$$

$M = 8$: $8^2 = 64$

$\rightsquigarrow 8 \cdot \underbrace{\log_2(8)}_{= 3 \text{ as } 2^3 = 8} = 24$

e.g.: $M = 4096$ ↗

DFT: 16 million cplx. mults

FFT: 50k cplx. mults

99,7% complexity reduction !!!