

Problem 8 (FFT of real and complex sequences)

Suppose that an FFT program is available that computes the DFT of a complex sequence. If we wish to compute the DFT of a real sequence, we may simply specify the imaginary part to be zero and use the program directly. However, the symmetry of the DFT of a real sequence can be used to reduce the amount of computation.

- (a) Let $x(n)$ be a real-valued sequence of length M , and let $X(\mu)$ be its DFT with real and imaginary parts denoted $X_R(\mu)$ and $X_I(\mu)$, respectively; i.e.,

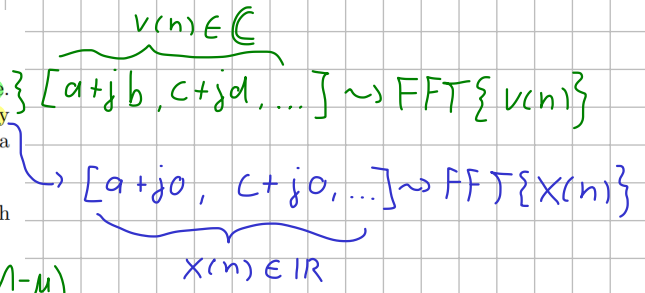
$$X(\mu) = X_R(\mu) + j X_I(\mu).$$

Show that if $x(n)$ is real, then $X_R(\mu) = X_R(M - \mu)$ and $X_I(\mu) = -X_I(M - \mu)$ for $\mu = 1, \dots, M - 1$.

Even symmetry

odd symmetry

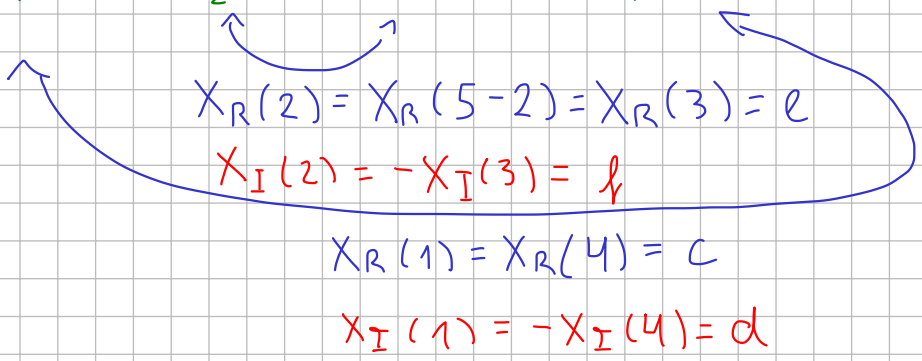
Hermitian symmetry / conjugate complex symmetry



Both: 2 variables } 1x Re
 value } 1x Im

$$X(\mu) = [a+jb, c+jd, e+jf, e-jf, c-jd]$$

$\mu: \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$



$$\text{FFTSHIFT}\{X(\mu)\} = [e-jf, c-jd, a+jb, c+jd, e+jf]$$

$-2\pi z \quad -\pi z \quad 0\pi z \quad \pi z \quad 2\pi z$

$$X(n) \in \mathbb{R} : X(n) = X^*(n)$$

$$\hookrightarrow X(\mu) = \sum_{n=0}^{M-1} X^*(n) e^{-j\mu \frac{2\pi}{M} \cdot n}$$

$$X(\mu) = \left(\sum_{n=0}^{M-1} X(n) e^{+j\mu \frac{2\pi}{M} \cdot n} \right)^* \quad \left\{ \begin{array}{l} \text{mult with } 1 \\ \hookrightarrow e^{-j\frac{2\pi}{M} \cdot M \cdot n} \end{array} \right.$$

$$X(\mu) = \left(\sum_{n=0}^{M-1} X(n) e^{+j\mu \frac{2\pi}{M} \cdot n} \cdot e^{-j\frac{2\pi}{M} \cdot M \cdot n} \right)^* \quad \left\{ \begin{array}{l} = e^{-j2\pi \cdot n} \\ = 1 \end{array} \right.$$

$$X(\mu) = \left(\sum_{n=0}^{M-1} X(n) e^{-j \frac{2\pi}{M} \cdot n (M-\mu)} \right)^*$$

$$= X^*(M-\mu) \rightsquigarrow X_R(\mu) = X_R(M-\mu) \ \& \ X_I(\mu) = -X_I(M-\mu)$$

(b) Now consider two real-valued sequences $x_1(n)$ and $x_2(n)$ with DFTs $X_1(\mu)$ and $X_2(\mu)$, respectively. Let $g(n)$ be the complex sequence $g(n) = x_1(n) + jx_2(n)$, with corresponding DFT $G(\mu) = G_R(\mu) + jG_I(\mu)$. Also, let $G_{OR}(\mu)$, $G_{ER}(\mu)$, $G_{OI}(\mu)$ and $G_{EI}(\mu)$ denote, respectively, the odd part of the real part, the even part of the real part, the odd part of the imaginary part, and the even part of the imaginary part of $G(\mu)$. Specifically, for $1 \leq \mu \leq M-1$,

$$G_{OR}(\mu) = 1/2\{G_R(\mu) - G_R(M-\mu)\},$$

$$G_{ER}(\mu) = 1/2\{G_R(\mu) + G_R(M-\mu)\},$$

$$G_{OI}(\mu) = 1/2\{G_I(\mu) - G_I(M-\mu)\},$$

$$G_{EI}(\mu) = 1/2\{G_I(\mu) + G_I(M-\mu)\},$$

and $G_{OR}(0) = G_{OI}(0) = 0$, $G_{ER}(0) = G_R(0)$, $G_{EI}(0) = G_I(0)$. Determine expressions for $X_1(\mu)$ and $X_2(\mu)$ in terms of $G_{OR}(\mu)$, $G_{ER}(\mu)$, $G_{OI}(\mu)$ and $G_{EI}(\mu)$.

$$g(n) = X_1(n) + j X_2(n) = X_{1E}(n) + X_{1O}(n) + j(X_{2E} + X_{2O})$$

$$= g_{re,even}(n) + g_{re,odd}(n) + j(g_{im,even}(n) + g_{im,odd}(n))$$



$$g(\mu) = \underbrace{X_1(\mu)}_{G_R(\mu)} + j \underbrace{X_2(\mu)}_{G_I(\mu)}$$

$$v(n) = v_{re,even}(n) + v_{re,odd}(n) + j v_{im,even}(n) + j v_{im,odd}(n)$$

$$V(e^{j\Omega}) = V_{re,even}(e^{j\Omega}) + V_{re,odd}(e^{j\Omega}) + j V_{im,even}(e^{j\Omega}) + j V_{im,odd}(e^{j\Omega})$$

$$= (X_{1,RE}(\mu) + j X_{1,IO}(\mu)) + j (X_{2,RE}(\mu) + j X_{2,IO}(\mu))$$

$$= \underbrace{X_{1,RE}(\mu) - X_{2,IO}(\mu)}_{\text{Real part}} + j \underbrace{(X_{1,IO}(\mu) + X_{2,RE}(\mu))}_{\text{imag. part}}$$

Real part : $G_R(\mu)$

imag. part : $G_I(\mu)$

$$\begin{aligned}
 g_{RE}(\mu) &= \frac{1}{2} \{ g_R(\mu) + g_R(M-\mu) \} \\
 &= \frac{1}{2} \{ X_{1,RE}(\mu) - \cancel{X_{2,IO}(\mu)} + \underbrace{X_{1,RE}(M-\mu)}_{= X_{1,RE}(\mu)} - \underbrace{\cancel{X_{2,IO}(M-\mu)}}_{= + X_{2,IO}(\mu)} \} \\
 &= \frac{1}{2} \{ 2 \cdot X_{1,RE}(\mu) \} = X_{1,RE}(\mu)
 \end{aligned}$$

$$g_{RO}(\mu) = \frac{1}{2} \{ g_R(\mu) - g_R(M-\mu) \} = -X_{2,IO}(\mu)$$

$$g_{IE} = \frac{1}{2} \{ g_I(\mu) + g_I(M-\mu) \} = X_{2,RE}(\mu)$$

$$g_{IO} = \frac{1}{2} \{ g_I(\mu) - g_I(M-\mu) \} = X_{1,IO}(\mu)$$

Finally:

$$\begin{aligned}
 X_1(\mu) &= X_{1,RE}(\mu) + j X_{1,IO}(\mu) \\
 &= g_{RE}(\mu) + j g_{IO}(\mu)
 \end{aligned}$$

$$\begin{aligned}
 X_2(\mu) &= X_{2,RE}(\mu) + j X_{2,IO}(\mu) \\
 &= g_{IE}(\mu) - j g_{RO}(\mu)
 \end{aligned}$$