Problem 8 **(FFT of real and complex sequences)**

Suppose that an FFT program is available that computes the DFT of a complex sequence. If we wish to compute the DFT of a real sequence, we may simply specify the imaginary part to be zero and use the program directly. However, the symmetry of the DFT of a real sequence can be used to reduce the amount of computation.

$\overbrace{[a+jb,\ c+jd,\ \dots]}^{v(n)\in\mathbb{C}} \rightsquigarrow \mathrm{FFT}\{v(n)\}$

$\underbrace{[a+j0,\ c+j0,\ \dots]}_{x(n)\in\mathbb{R}} \rightsquigarrow \mathrm{FFT}\{x(n)\}$

(a) Let $x(n)$ be a real-valued sequence of length M, and let $X(\mu)$ be its DFT with real and imaginary parts denoted $X_R(\mu)$ and $X_I(\mu)$, respectively; i.e.,

$$X(\mu) = X_R(\mu) + j\,X_I(\mu).$$

Show that if $x(n)$ is real, then $\underbrace{X_R(\mu) = X_R(M-\mu)}_{\text{even symmetry}}$ and $\underbrace{X_I(\mu) = -X_I(M-\mu)}_{\text{odd symmetry}}$ for $\mu = 1, \dots, M-1$.

$X_I^*(M-\mu)$

Both: $\dfrac{\text{2 variables}}{\text{value}} \Big\}$ 1x Re, 1x Im

↳ Hermitian symmetry / Conjugate complex symmetry

$$X(\mu) = [\,a+jb,\quad c+jd \quad e+jf \quad e-jf \quad c-jd\,]$$

μ: 0 1 2 3 4

$X_R(2) = X_R(5-2) = X_R(3) = e$

$X_I(2) = -X_I(3) = f$

$X_R(1) = X_R(4) = c$

$X_I(1) = -X_I(4) = d$

$$\mathrm{FFTSHIFT}\{X(\mu)\} = [\ \underset{-2\,\mathrm{Hz}}{e-jf} \quad \underset{-1\,\mathrm{Hz}}{c-jd} \quad \underset{0\,\mathrm{Hz}}{a+jb}, \quad \underset{1\,\mathrm{Hz}}{c+jd} \quad \underset{2\,\mathrm{Hz}}{e+jf}\ ]$$

$$x(n) \in \mathbb{R}: \quad x(n) = x^*(n)$$

$$\hookrightarrow \quad X(\mu) = \sum_{n=0}^{M-1} x^*(n)\, e^{-j\mu \frac{2\pi}{M} \cdot n}$$

$$X(\mu) = \left(\sum_{n=0}^{M-1} x(n)\, e^{+j\mu \frac{2\pi}{M} \cdot n} \right)^*$$

mult with 1

$\hookrightarrow e^{-j \frac{2\pi}{M} \cdot M \cdot n} = e^{-j 2\pi \cdot n} = 1$

$$X(\mu) = \left(\sum_{n=0}^{M-1} x(n)\, e^{+j\mu \frac{2\pi}{M} \cdot n} \cdot e^{-j \frac{2\pi}{M} \cdot M \cdot n} \right)^*$$

$$X(\mu) = \left(\sum_{n=0}^{M-1} x(n)\, e^{-j \frac{2\pi}{M} \cdot n (M-\mu)} \right)^*$$

$$= X^*(M-\mu) \rightsquigarrow X_R(\mu) = X_R(M-\mu) \ \& \ X_I(\mu) = -X_I(M-\mu)$$

(b) Now consider two real-valued sequences $x_1(n)$ and $x_2(n)$ with DFTs $X_1(\mu)$ and $X_2(\mu)$, respectively. Let $g(n)$ be the complex sequence $g(n) = x_1(n) + j\,x_2(n)$, with corresponding DFT $G(\mu) = G_R(\mu) + j\,G_I(\mu)$. Also, let $G_{OR}(\mu)$, $G_{ER}(\mu)$, $G_{OI}(\mu)$ and $G_{EI}(\mu)$ denote, respectively, the odd part of the real part, the even part of the real part, the odd part of the imaginary part, and the even part of the imaginary part of $G(\mu)$. Specifically, for $1 \leq \mu \leq M-1$,

$$G_{OR}(\mu) = 1/2\{G_R(\mu) - G_R(M-\mu)\},$$
$$G_{ER}(\mu) = 1/2\{G_R(\mu) + G_R(M-\mu)\},$$
$$G_{OI}(\mu) = 1/2\{G_I(\mu) - G_I(M-\mu)\},$$
$$G_{EI}(\mu) = 1/2\{G_I(\mu) + G_I(M-\mu)\},$$

and $G_{OR}(0) = G_{OI}(0) = 0$, $G_{ER}(0) = G_R(0)$, $G_{EI}(0) = G_I(0)$. Determine expressions for $X_1(\mu)$ and $X_2(\mu)$ in terms of $G_{OR}(\mu)$, $G_{ER}(\mu)$, $G_{OI}(\mu)$ and $G_{EI}(\mu)$.

$$g(n) = x_1(n) + j\,x_2(n) = x_{1E}(n) + x_{1O}(n) + j(x_{2E} + x_{2O})$$
$$= g_{re,even}(n) + g_{re,odd}(n)$$
$$+ j(g_{im,even}(n) + g_{im,odd}(n))$$

$$v(n) = v_{\text{re,even}}(n) + v_{\text{re,odd}}(n) + j\,v_{\text{im,even}}(n) + j\,v_{\text{im,odd}}(n)$$

$$V(e^{j\Omega}) = V_{\text{re,even}}(e^{j\Omega}) + V_{\text{re,odd}}(e^{j\Omega}) + j\,V_{\text{im,even}}(e^{j\Omega}) + j\,V_{\text{im,odd}}(e^{j\Omega}).$$

$$G(\mu) = \underbrace{X_1(\mu)}_{G_R(\mu)} + j\,\underbrace{X_2(\mu)}_{G_I(\mu)}$$

$$= (X_{1,RE}(\mu) + j\,X_{1,IO}(\mu)) + j\,(X_{2,RE}(\mu) + j\,X_{2,IO}(\mu))$$

$$= \underbrace{X_{1,RE}(\mu) - X_{2,IO}(\mu)}_{\text{real part}: \ G_R(\mu)} + \underbrace{j(X_{1,IO}(\mu) + X_{2,RE}(\mu))}_{\text{imag. part}: \ G_I(\mu)}$$

$$g_{RE}(\mu) = \frac{1}{2}\left\{ g_R(\mu) + g_R(M-\mu) \right\}$$

$$= \frac{1}{2}\left\{ X_{1,RE}(\mu) - \cancel{X_{2,IO}(\mu)} + \underbrace{X_{1,RE}(M-\mu)}_{=X_{1,RE}(\mu)} - \underbrace{\cancel{X_{2,IO}(M-\mu)}}_{=+X_{2,IO}(\mu)} \right\}$$

$$= \frac{1}{2}\left\{ 2 \cdot X_{1,RE}(\mu) \right\} = X_{1,RE}(\mu)$$

$$g_{RO}(\mu) = \frac{1}{2}\left\{ g_R(\mu) - g_R(M-\mu) \right\} = -X_{2,IO}(\mu)$$

$$g_{IE} = \frac{1}{2}\left\{ g_I(\mu) + g_I(M-\mu) \right\} = X_{2,RE}(\mu)$$

$$g_{IO} = \frac{1}{2}\left\{ g_I(\mu) - g_I(M-\mu) \right\} = X_{1,IO}(\mu)$$

Finally: $X_1(\mu) = X_{1,RE}(\mu) + j\, X_{1,IO}(\mu)$

$$= g_{RE}(\mu) + j\, g_{IO}(\mu)$$

$$X_2(\mu) = X_{2,RE}(\mu) + j\, X_{2,IO}(\mu)$$

$$= g_{IE}(\mu) - j\, g_{RO}(\mu)$$