ADSP - Ex. 7

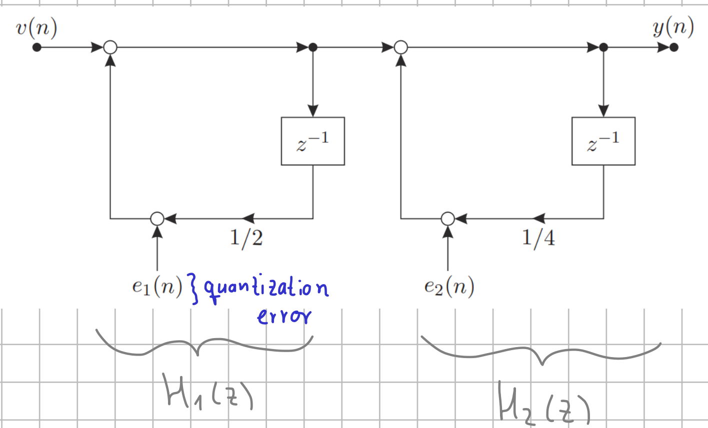
Problem 11 (round-off effects in digital filters)

Determine the variance of the round-off noise at the output of the two cascade realizations of the filter with system function

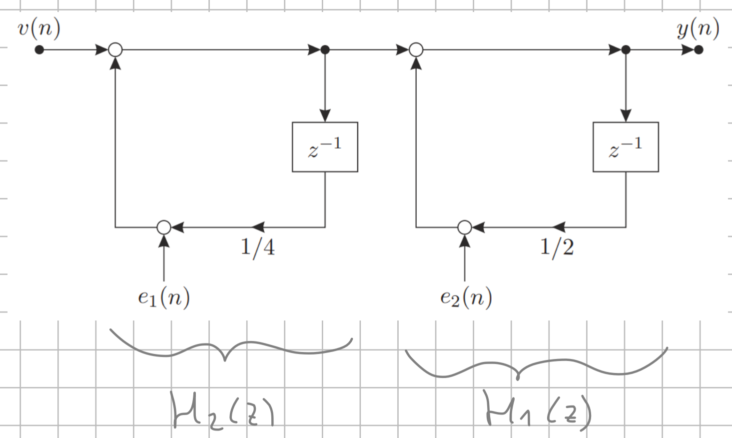
$$H(z) = H_1(z) \cdot H_2(z) \tag{1}$$

$$H_1(z) = \frac{1}{1-0,5z^{-1}}, H_2(z) = \frac{1}{1-0,25z^{-1}} \tag{2}$$

I. $H(z) = H_1(z) \cdot H_2(z)$



II. $H(z) = H_2(z) \cdot H_1(z)$



$$H(z) = H_1(z) \cdot H_2(z) = \frac{1}{1-0,5\cdot z^{-1}} \cdot \frac{1}{1-0,25\cdot z^{-1}}$$

$$= \left.\frac{1}{1-0,75\cdot z^{-1}+0,125\cdot z^{-2}}\right\} \cdot \frac{z^2}{z^2}$$

$$= \frac{z^2}{z^2-0,75z+0,125}$$

Polynom division:

$$z^2 : z^2-0,75z+0,125 = 1 + \frac{0,75z-0,125}{z^2-0,75z+0,125}$$

$$\frac{z^2}{z^2} = 1$$

$$-(z^2-0,75z+0,125)$$

$$z^2-(z^2-0,75z+0,125) \rightsquigarrow \underbrace{0,75z-0,125}_{deg=1}$$

$$= 1 + \frac{0,75z-0,125}{(z-0,5)(z-0,25)}$$

Stop if degree of rest < degree of divisor ($=2$)

Partial fraction expansion

$$= 1 + \frac{0,75z-0,125}{(z-0,5)(z-0,25)} = 1 + \frac{A}{(z-0,5)} + \frac{B}{(z-0,25)}$$

$$\frac{0,75z-0,125}{(z-0,5)(z-0,25)}(z-0,5)(z-0,25) = \frac{A}{(z-0,5)}(z-0,5)(z-0,25) + \frac{B}{(z-0,25)}(z-0,5)(z-0,25)$$

$$\frac{0{,}75z-0{,}125}{\cancel{(z-0{,}5)(z-0{,}25)}}\cancel{(z-0{,}5)(z-0{,}25)} = \frac{A}{\cancel{(z-0{,}5)}}\cancel{(z-0{,}5)}(z-0{,}25)+\frac{B}{\cancel{(z-0{,}25)}}(z-0{,}5)\cancel{(z-0{,}25)}$$

$$0{,}75\cdot z-0{,}125 = A\cdot(z-0{,}25)+B\cdot(z-0{,}5)$$

Choose z to cancel a term: $\hookrightarrow z=0{,}25$

$\hookrightarrow 0{,}75\cdot 0{,}25-0{,}125 = B\cdot(-0{,}25)$

$\rightsquigarrow B=-0{,}25$

$$0{,}75\cdot z-0{,}125 = A\cdot(z-0{,}25)+B\cdot(z-0{,}5)$$

$\hookrightarrow z=0{,}5$

$\downarrow$

$0{,}75\cdot 0{,}5-0{,}125 = 0{,}25\cdot A$

$\rightsquigarrow A=1$

$$= z^{-1}\cdot\frac{1}{1-0{,}5\cdot z^{-1}} \Big\} = -0{,}25\cdot z^{-1}\frac{1}{1-0{,}25\cdot z^{-1}}$$

$$\rightsquigarrow H(z) = 1+\frac{1}{(z-0{,}5)}+\frac{-0{,}25}{(z-0{,}25)}$$

$$\gamma_0(n)+\underbrace{(0{,}5)^{n-1}}_{0{,}5^n\cdot\underbrace{0{,}5^{-1}}_{=1/0{,}5=2}}\cdot\gamma_{-1}(n-1)-0{,}25\cdot\underbrace{(0{,}25)^{n-1}}_{0{,}25^n\cdot 4}\cdot\gamma_{-1}(n-1)$$

$$h(n) = \gamma_0(n)+\underbrace{\gamma_{-1}(n-1)}_{=\gamma_{-1}(-1)=0\text{ for }n=0}\cdot(2\cdot 0{,}5^n-0{,}25^n)$$

$n=0$: $h(0) = 1 + 0$

$h(1) = 0 + 1\cdot(2\cdot 0{,}5-0{,}25) = 0{,}75$

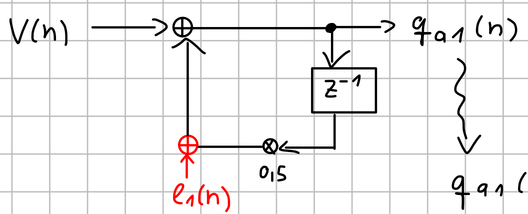
connect $\gamma_0(n)$ & $\gamma_{-1}(n-1)$ to $\gamma_{-1}(n)$

$\Downarrow$

$$h(n) = \gamma_{-1}(n)\cdot(2\cdot 0{,}5^n-0{,}25^n)$$

Quantization error for I.

$\hookrightarrow$ Modelled as additive noise component, uncorrelated to signal $v(n)$



$$q_{a1}(n) = V(n)+e_1(n)+0{,}5\cdot q_{a1}(n-1)$$

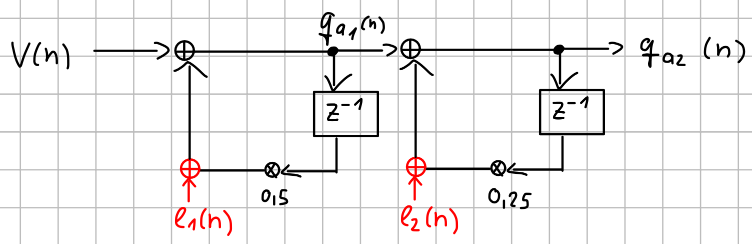
$$Q_{a1}(z) = V(z)+E_1(z)+0{,}5\cdot Q_{a1}(z)\cdot z^{-1}$$

$$= \underbrace{\frac{1}{1-0{,}5\cdot z^{-1}}\cdot\left(V(z)+E_1(z)\right)}$$

As $v(n)$ & $e_1(n)$ uncorrelated: Split term into signal & noise component

$$Q_{a1}(z) = E_1(z)\cdot\frac{1}{1-0{,}5\cdot z^{-1}}$$

$$q_{a_2}(n) = q_{a_1}(n) + e_2(n) + 0{,}25 \cdot q_{a_2}(n-1)$$



$$Q_{a_2}(z) = Q_{a_1}(z) + E_2(z) + 0{,}25 \cdot Q_{a_2}(z) \cdot z^{-1}$$

$$= \frac{1}{1 - 0{,}25 \cdot z^{-1}} \left(Q_{a_1}(z) + E_2(z) \right)$$

$$= \underbrace{\frac{E_1(z)}{(1-0{,}25 \cdot z^{-1})(1-0{,}5 \cdot z^{-1})}}_{\text{Subsystem 1 noise}} + \underbrace{\frac{E_2(z)}{(1-0{,}25 \cdot z^{-1})}}_{\text{Subsystem 2 noise}}$$

$H(z) \left\{ (1-0{,}25 \cdot z^{-1})(1-0{,}5 \cdot z^{-1}) \right.$ $\qquad$ $\left. (1-0{,}25 \cdot z^{-1}) \right\} H_2(z)$

Subsystem 1 noise propagated through whole system

Quantization noise variance: $\sigma^2 = \underbrace{\sigma_e^2}_{\text{variance of } e_{1/2}(n)} \cdot \sum_{n=-\infty}^{\infty} h^2(n)$

$$\hookrightarrow \sigma^2_{qa1} = \sigma_e^2 \cdot \left(\sum_{n=0}^{\infty} h^2(n) + \sum_{n=0}^{\infty} h_2^2(n) \right)$$

Start at 0 due to $\gamma_{-1}(n)$

$$\sum_{n=0}^{\infty} h_2^2(n) = \underbrace{\sum_{n=0}^{\infty} 0{,}25^{n \cdot 2} = \frac{1}{1-0{,}25^2}}_{\text{Geometric series: } \sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \text{ for } |q|<1} = \frac{16}{15}$$

$$\sum_{n=0}^{\infty} h_1^2(n) = \sum_{n=0}^{\infty} 0{,}5^{2n} = \frac{1}{1-0{,}5^2} = \frac{4}{3}$$

$$\sum_{n=0}^{\infty} h^2(n) = \sum_{n=0}^{\infty} (2 \cdot 0{,}5^n - 0{,}25^n)^2 = \sum_{n=0}^{\infty} 4 \cdot 0{,}5^{2n} - 2^2 \cdot \underbrace{0{,}5^n \cdot 0{,}25^n}_{(0{,}5 \cdot 0{,}25)^n = \left(\frac{1}{8}\right)^n} + 0{,}25^{2n}$$

$$\hookrightarrow = \frac{4}{1-\frac{1}{4}} - \frac{4}{1-\frac{1}{8}} + \frac{1}{1-\frac{1}{16}} = \frac{64}{35} \approx 1{,}829$$

I. $$\sigma^2_{qa1} = \sigma_e^2 \cdot \left(\sum_{n=0}^{\infty} h^2(n) + \sum_{n=0}^{\infty} h_2^2(n) \right) = \sigma_e^2 \cdot \left(\frac{64}{35} + \frac{16}{15} \right) = \frac{304}{105} \sigma_e^2 \approx 2{,}895 \cdot \sigma_e^2$$

II. $$\sigma^2_{qa2} = \sigma_e^2 \cdot \left(\sum_{n=0}^{\infty} h^2(n) + \sum_{n=0}^{\infty} h_1^2(n) \right) = \sigma_e^2 \cdot \left(\frac{64}{35} + \frac{4}{3} \right) = \frac{332}{105} \cdot \sigma_e^2 \approx 3{,}162 \cdot \sigma_e^2$$

$$\rightsquigarrow \frac{\sigma^2_{qa2}}{\sigma^2_{qa1}} \approx 1{,}0921$$

↳ cascade II has a 9% higher noise variance than cascade I.