ADSP- Ex. 9

Problem 13 (Digital IIR Filter Design)

The system function of a discrete-time system is

$$H(z) = \frac{2}{1 - e^{-0.2}z^{-1}} - \frac{1}{1 - e^{-0.4}z^{-1}}.$$

(a) Assume that this discrete-time filter was designed by the impulse invariance method with $T = 2$, i.e. $h_i = h_a(iT)$, where $h_a(t)$ is real. Find the system function $H_a(s)$ of a continuous-time filter that could have been the basis for the design. Is your answer unique? If not, find another system function $H_a(s)$.

Impulse invariance method:

Convert an analog filter designed in the analog domain (s-domain) to a digital filter (z-domain)

analog filter impulse response: $h_a(t)$ → sample with $f_s = \frac{1}{T}$ at $\boxed{h_a(nT) = h_n(n)}$

$$h_a(t) = \begin{cases} \sum_{i=0}^{N-1} A_i \cdot e^{s_{\infty,i} \cdot t} & , t \geq 0 \\ 0 & , \text{else} \end{cases}$$

Laplace transform

$$H_a(s) = \sum_{i=0}^{N-1} \frac{A_i}{\underbrace{s - s_{\infty,i}}_{\text{distinct poles}}}$$

A_i } partial fraction expansion term (to get distinct poles)

$$h_n(n) = h_a(nT) = \sum_{i=0}^{N-1} A_i \cdot e^{s_{\infty,i} \cdot nT}$$

z-transform

$$H(z) = \sum_{i=0}^{N-1} \frac{A_i}{1 - \underbrace{e^{s_{\infty,i} \cdot T}}\cdot z^{-1}}$$

$$\boxed{\frac{1}{s - s_{\infty,i}} \longleftrightarrow \frac{1}{1 - e^{s_{\infty,i} \cdot T} \cdot z^{-1}}}$$

$$\underbrace{\boxed{z_{\infty,i} = e^{s_{\infty,i} \cdot T}}}_{\text{Poles of digital IIR-filter}}$$

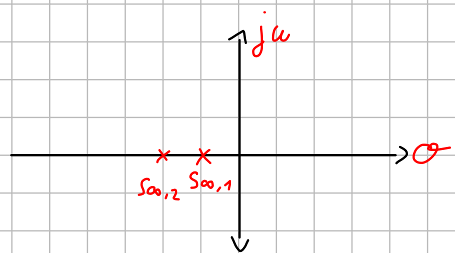
Here:

$$H(z) = \frac{2}{1 - \underbrace{e^{-0,2}}_{z_{\infty,0}} z^{-1}} - \frac{1}{1 - \underbrace{e^{-0,4}}_{z_{\infty,1}} \cdot z^{-1}}$$

$$z_{\infty,i} = e^{s_{\infty,i} \cdot T}$$

$\hookrightarrow -0,2 = s_{\infty,0} \cdot \underbrace{T}_{=2}$

$$s_{\infty,0} = \frac{-0,2}{2} = -0,1 \qquad s_{\infty,1} = -0,2$$

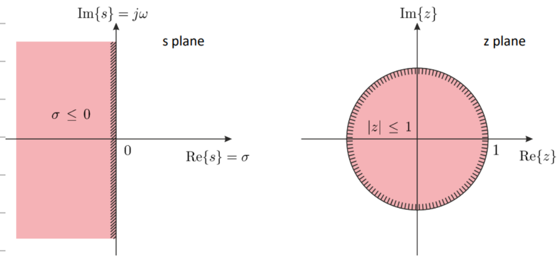


2 distinct poles on σ-axis
↳ unique solution

$$\hookrightarrow H_a(s) = \sum_{i=0}^{1} \frac{A_i}{s - s_{\infty,i}} = \frac{2}{s - (-0,1)} - \frac{1}{s - (-0,2)}$$

$$= \frac{2}{s + 0,1} - \frac{1}{s + 0,2}$$

(b) Assume that $H(z)$ was obtained by the bilinear transform with $T = 2$. Find the system function $H_a(s)$ that could have been the basis for the design. Is your answer unique? If not, find another $H_a(s)$.



Bilinear transform: Algebraic transform from s- to z-domain

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right), \quad z = \frac{1+\frac{T}{2}\cdot s}{1-\frac{T}{2}\cdot s}$$

$$H(z) = H_a\left[s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$

$$H_a(s) = H\left[z = \frac{1+\frac{T}{2}\cdot s}{1-\frac{T}{2}\cdot s}\right]$$

Bilinear transform is unique! $j\omega$-axis maps onto unit circle

$$\hookrightarrow H_a(s) = \frac{2}{1-e^{-0,2}\left[\frac{1+\frac{T}{2}\cdot s}{1-\frac{T}{2}\cdot s}\right]^{-1}} - \frac{1}{1-e^{-0,4}\left[\frac{1+\frac{T}{2}\cdot s}{1-\frac{T}{2}\cdot s}\right]^{-1}}$$

$T = 2 \rightsquigarrow \frac{2}{2} = 1$

$$= \frac{2}{1-e^{-0,2}\left(\frac{1-s}{1+s}\right)} - \frac{1}{1-e^{-0,4}\left(\frac{1-s}{1+s}\right)}$$

$$\cdot \frac{1+s}{1+s} \quad = \frac{2(1+s)}{(1+s)-e^{-0,2}\cdot(1-s)} - \frac{1+s}{(1+s)-e^{-0,4}\cdot(1-s)}$$

factorize $(1+e^{-0,2})$ to get $s + \dots$

$$= \frac{2(1+s)}{s(1+e^{-0,2})+(1-e^{-0,2})} - \frac{1+s}{s(1+e^{-0,4})+(1-e^{-0,4})}$$

$$= \underbrace{\frac{2}{1+e^{-0,2}}}_{\text{scalar factor}} \cdot \underbrace{\frac{s+1}{s+\frac{1-e^{-0,2}}{1+e^{-0,2}}}}_{\text{term with } s} - \frac{1}{1+e^{-0,4}} \cdot \frac{s+1}{s+\frac{1-e^{-0,4}}{1+e^{-0,4}}}$$

Due to the property of the unique transform of the bilinear transform, the representation is unique

Problem 14 (Digital IIR Filter Design)

A discrete-time lowpass filter is to be designed by applying the impulse invariance method to a continuous-time Butterworth filter having magnitude-squared function

$$|H(j\omega)|^2 = \frac{1}{1+(\frac{\omega}{\omega_{cut}})^{2N}}$$

The specifications for the discrete-time signal are

$$0.89125 \le |H(e^{j\Omega})| \le 1, \qquad 0 \le |\Omega| \le 0.2\pi,$$
$$|H(e^{j\Omega})| \le 0.17783, \qquad 0.3\pi \le |\Omega| \le \pi.$$

Assume that aliasing will not be a problem, i.e., design the continuous-time Butterworth filter to meet passband and stopband specifications as determined by the discrete-time filter.

(a) Sketch the tolerance bounds on the magnitude of the frequency response, $|H(j\omega)|$, of the continuous-time Butterworth filter such that after application of the impulse invariance method, the resulting discrete-time filter will satisfy the given design specifications. Do not assume that $T = 1$.

Filter specifications

For discrete-time signal $\xrightarrow{n \to \Omega}$ DTFT

$$0.89125 \le |H(e^{j\Omega})| \le 1, \qquad 0 \le |\Omega| \le 0.2\pi,$$
$$|H(e^{j\Omega})| \le 0.17783, \qquad 0.3\pi \le |\Omega| \le \pi.$$

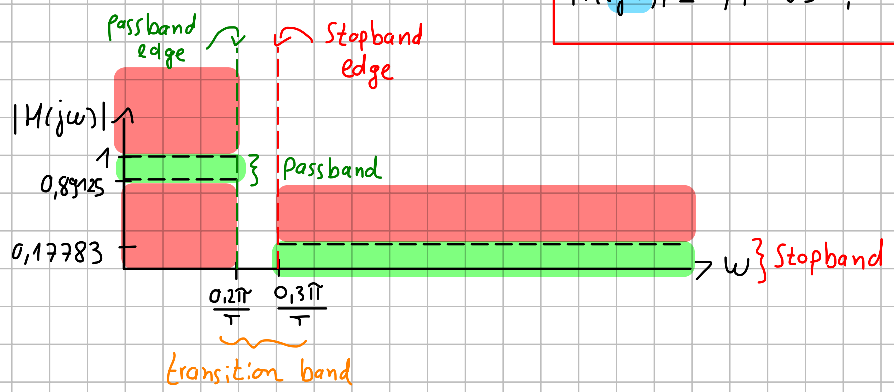
$\Omega \to \omega$

$\Omega = \omega T = \frac{\omega}{f_s}$

$\hookrightarrow \omega = \frac{\Omega}{T} = \Omega \cdot f_s$

continuous-time Butterworth filter $t \to \omega$

$$\boxed{\begin{array}{ll} 0{,}89125 \le |H(j\omega)| \le 1, & 0 \le |\omega| \le \frac{0{,}2\pi}{T} \\ |H(j\omega)| \le 0{,}17783, & \frac{0{,}3\pi}{T} \le |\omega| \le \frac{\pi}{T} \end{array}}$$

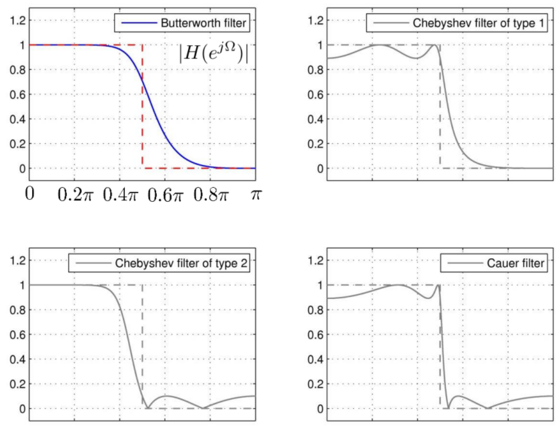


Butterworth filters

Lowpass Butterworth filters are allpole-filters characterized by the squared magnitude frequency response

$$|H(\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_{cut}}\right)^{2N}},$$

N is the order of the filter,
ω_{cut} is the -3 dB frequency (cut-off frequency).



(b) Determine the integer order N and the quantity $\omega_{cut}T$ such that the continuous-time Butterworth filter exactly meets the specifications determined in part (a) at the passband edge.

At Stopband edge: $\frac{1}{1+\left(\frac{\omega_{stop}}{\omega_{cut}}\right)^{2N}} = \delta_2^2 \rightsquigarrow N = \frac{\log_{10}\left([1/\delta_2^2]-1\right)}{2\log_{10}(\omega_{stop}/\omega_{cut})}$

Eq. I.

Stopband edge: $\underbrace{|H(j0{,}3\pi/T)|^2}_{a} \overset{!}{=} 0{,}17783^2$

$$= \frac{1}{1+\underbrace{\left(\frac{0{,}3\pi}{\omega_c T}\right)}_{\frac{b}{c}}^{2N}} \qquad a = \frac{1}{1+\left(\frac{b}{c}\right)^{2N}}$$

$$a = \frac{1}{1+\left(\frac{b}{c}\right)^{2N}}$$

$$\frac{1}{a} = 1+\left(\frac{b}{c}\right)^{2N}$$

$$\log_{10}\left(\frac{1}{a}-1\right) = 2\cdot N\cdot \log_{10}(b/c)$$

$$N = \frac{\log_{10}\left(\frac{1}{a}-1\right)}{2\cdot\log_{10}(b/c)} = \frac{\log_{10}\left(\frac{1}{0{,}17783^2}-1\right)}{2\cdot\log_{10}\left(\frac{0{,}3\pi}{\omega_c T}\right)}$$

Eq. II

passband edge: $\underbrace{|H(j0{,}2\pi/T)|^2}_{d} \overset{!}{=} 0{,}89125^2 = \frac{1}{1+\underbrace{\left(\frac{0{,}2\pi}{\omega_c T}\right)}_{\frac{\omega}{\omega_c}=\frac{e}{c}}^{2N}}$

$$d = \frac{1}{1+\left(\frac{e}{c}\right)^{2N}}$$

$$d^{-1}-1 = \left(\frac{e}{c}\right)^{2N}$$

$$\log_{10}(d^{-1}-1) = 2N\cdot(\log_{10}e - \log_{10}c)$$

$$\log_{10}c = \log_{10}e - \frac{\log_{10}(d^{-1}-1)}{2N}$$

$$c = 10^{\left(\log_{10}e - \frac{\log_{10}(d^{-1}-1)}{2N}\right)}$$

III.) $$c = \omega_c T = 10^{\left(\log_{10}(0{,}2\pi) - \frac{\log_{10}\left(\frac{1}{0{,}89125^2}-1\right)}{2N}\right)}$$

↳ cut-off. freq. specification of cont. time filter at passband edge

~> solve system of equations I.) & II.) for $\underbrace{N}_{1.)}$ & $\underbrace{w_c \cdot T}_{2.)}$

2 unknowns

↳ $w_c \cdot T = 0{,}70474$

$N \approx 5{,}88$

↳ round up to next integer

↳ $N = 6$

↳ insert into III.)

↳ $\underbrace{w_c \cdot T = 0{,}7032 \text{ for } N = 6}_{\text{fulfills our filter specifications at the passband edge}}$