TECHNISCHE FAKULTÄT DER
CHRISTIAN-ALBRECHTS-UNIVERSITÄT
ZU KIEL

DIGITALE
SIGNALVERARBEITUNG UND
SYSTEMTHEORIE

Signals and Systems II

Exam WS 2022/2023

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Date:	03.03.2023
Name:	______________________
Matriculation Number:	______________________

Declaration of the candidate before the start of the examination

I hereby confirm that I am registered for, authorized to sit and eligible to take this examination.
I understand that the date for inspecting the examination will be announced by the EE&IT Examination Office, as soon as my provisional examination result has been published in the QIS portal. After the inspection date, I am able to request my final grade in the QIS portal. I am able to appeal against this examination procedure until the end of the period for academic appeals for the second examination period at the CAU. After this, my grade becomes final.

Signature: ______________________

Marking

Problem	1	2	3
Points	/33.5	/32.5	/34

Total number of points: ______________ /100

Inspection/Return

I hereby confirm that I have acknowledged the marking of this examination and that I agree with the marking noted on this cover sheet.

☐ The examination papers will remain with me. Any later objection to the marking or grading is no longer possible.

Kiel, dated ______________ Signature: ______________________

Signals and Systems II

Exam WS 2022/2023

Examiner:	Prof. Dr.-Ing. Gerhard Schmidt
Room:	OS75 - Hans-Heinrich-Driftmann-Hörsaal
Date:	03.03.2023
Begin:	09:00 h
Reading Time:	10 Minutes
Working Time:	90 Minutes

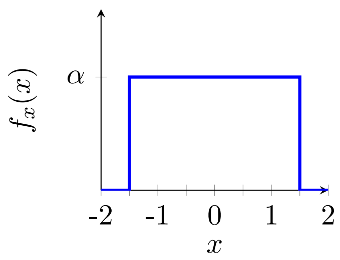
Hinweise

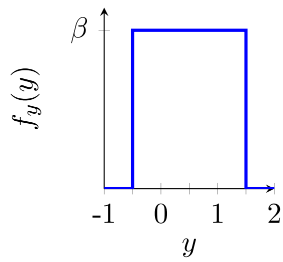
- Lay out your student or personal ID for inspection.
- Label **each** paper with your **name** and **matriculation number**. Please use a **new sheet of paper** for **each task**. Additional paper is available on request.
- Do **not** use **pencil or red pen**.
- All aids – except for those which allow the communication with another person – are allowed. Prohibited aids are to be kept out of reach and should be turned off.
- The direct communication with any person who is not part of the exam supervision team is prohibited.
- For full credit, your solution is required to be comprehensible and well-reasoned. All sketches of functions require proper labeling of the axes. Please understand that the shown point distribution is only preliminary!
- In case you should feel negatively impacted by your surroundings during the exam, you must notify an exam supervisor immediately.
- The imminent ending of the exam will be announced 5 minutes and 1 minute prior to the scheduled ending time. Once the **end of the exam** has been announced, you **must stop writing** immediately.
- At the end of the exam, put together all solution sheets and hand them to an exam supervisor together with the exam tasks and the **signed cover sheet**.
- Before all exams have been collected, you are prohibited from talking or leaving your seat. Any form of communication at this point in time will still be regarded as an **attempt of deception**.
- During the **reading time, working on the exam tasks is prohibited**. Consequently, all writing tools and other allowed aids should be set aside. Any violation of this rule will be considered as an **attempt of deception**.

Task 1 (33.5 Points)

Part 1 *This part of the task can be solved independently of parts 2 and 3.*

Let the real, statistically independent random variables x and y and their probability densities be given.

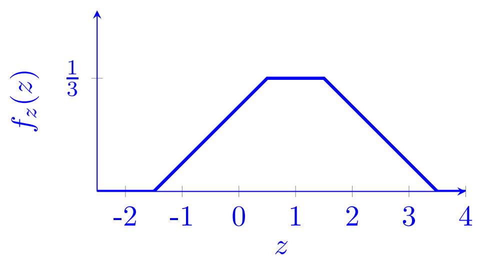




(a) Determine the constants α and β. (2 P)

$$\alpha = \frac{1}{3}, \beta = \frac{1}{2}$$

(b) Sketch the probability density of the mapping $z = x + y$. Label it! (4 P)



Let the real random variable α be given, which is uniformly distributed over the interval $[-\frac{3\pi}{2}, \frac{\pi}{2})$. In addition, let a complex deterministic mapping be given:

$$\beta(\alpha) = 3e^{-j\alpha} + \frac{1}{2}.$$

(c) Determine the probability density $f_\alpha(\alpha)$. (2 P)

$$f_\alpha(\alpha) = \begin{cases} \frac{1}{2\pi}, & \text{für } -\frac{3\pi}{2} \leq \alpha < \frac{\pi}{2} \\ 0, & \text{sonst.} \end{cases}$$

(d) Calculate the expectation value of the mapping of $\beta(\alpha)$ separately for real and ima- (5 P)

ginary part.

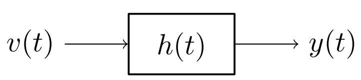
$$\mathrm{Re}\{\beta(\alpha)\} = 3\cos(\alpha) + \frac{1}{2}$$
$$\mathrm{Im}\{\beta(\alpha)\} = -3\sin(\alpha)$$

$$\begin{aligned} E\{\mathrm{Re}\{\beta(\alpha)\}\} &= \frac{1}{2\pi}\int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} 3\cos(\alpha) + \frac{1}{2}\, d\alpha \\ &= \frac{3}{2\pi}\left[\sin(\alpha) + \frac{1}{6}\alpha\right]_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{3}{2\pi}(1 + \frac{\pi}{12} - 1 + \frac{3\pi}{12}) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} E\{\mathrm{Im}\{\beta(\alpha)\}\} &= -\frac{3}{2\pi}\int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} \sin(\alpha)\, d\alpha \\ &= \frac{3}{2\pi}\left[\cos(\alpha)\right]_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} \\ &= 0 \end{aligned}$$

Part 2 *This part of the task can be solved independently of parts 1 and 3.*

Let the white, mean-free noise signal $v(t)$ be given. The signal is used as input signal for a linear, time-invariant transmission path, which can be described by $H(j\omega)$. Furthermore, let the autocorrelation function $s_{yy}(\tau)$ be known.



(e) Describe in words the autocorrelation and power density spectrum of the random process $v(t)$. (2 P)
Characteristic of white noise is a constant power density spectrum. The autocorrelation corresponds to the Dirac pulse.

(f) How would you proceed to determine the magnitude of the transfer function? Which relationships would you exploit? State the mathematical relationship! (3 P)
Exploit relationship between $s_{vv}(\tau)$, $s_{vy}(\tau)$ and $v(t), y(t)$:

$$S_{vy}(j\omega) = H(j\omega)S_{vv}(j\omega),$$

$$S_{yy}(j\omega) = H(j\omega)H^*(j\omega)S_{vv}(j\omega) = |H(j\omega)|^2 S_{vv}(j\omega).$$

Determine the power density spectra $S_{vv}(j\omega)$ and $S_{yy}(j\omega)$ and insert them:

$$|H(j\omega)| = \sqrt{\frac{|S_{yy}(j\omega)|}{|S_{vv}(j\omega)|}}.$$

Alternativly:

$$|H(j\omega)| = \sqrt{\frac{|S_{vy}(j\omega)|}{|S_{vv}(j\omega)|}}$$

or

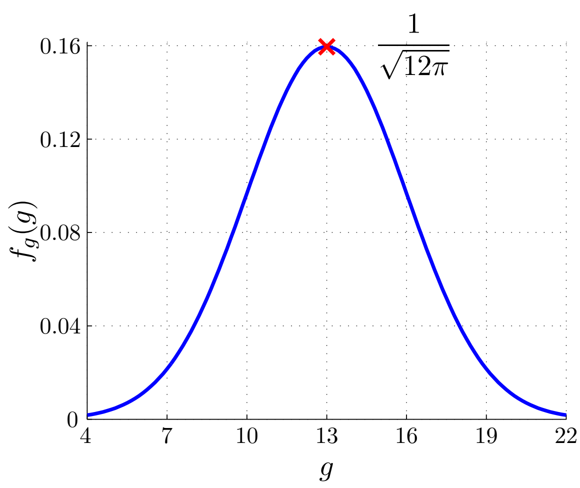
$$|H(j\omega)| = \sqrt{\frac{|S_{yv}(j\omega)|}{|S_{vv}(j\omega)|}}.$$

Let the signal $v(t)$ now be ideally band-limited, so that $S_{vv}(j\omega) = 0 \;\forall\, \omega \notin (-\omega_g, \omega_g)$ applies.

(g) Describe what changes occur in terms of autocorrelation and power density spectrum. (2 P)

Due to the band limitation, the spectrum is no longer constant with S_0, but a rectangular function of width $2\omega_g$ and value S_0. In the time domain, there is now a si-function instead of a Dirac pulse.

Now let the following Gaussian distribution $f_g(g)$ be given:



(h) Determine the 2nd moment of the distribution density function. (4.5 P)

$$m_g^{(2)} = \sigma_g^2 + |m_g|^2$$

m_g can be read: $m_g = 13$.
Using the Gaussian distribution

$$f_g(g) = \frac{1}{\sqrt{2\pi}\sigma_g} e^{\frac{-(g-m_g)^2}{2\sigma_g^2}}$$

follows

$$f_g(g = m_g) = \frac{1}{\sqrt{12\pi}},$$

therefore applies

$$\sigma_g^2 = 6.$$

Therefore, the following applies to the 2nd moment of the distribution density:

$$m_g^{(2)} = 6 + 13^2 = 175.$$

Part 3 *This part of the task can be solved independently of parts 1 and 2.*

Let the probability density $f_a(a)$ of the real and continuous random variable a be given by:

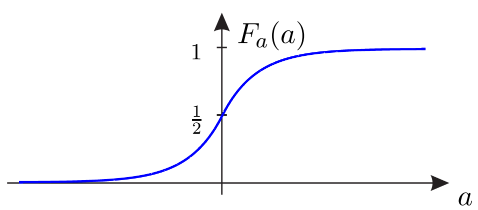
$$f_a(a) = \delta e^{-|a|\phi^{-1}}.$$

For the parameters δ and ϕ the following applies: $\delta, \phi > 0$.

(i) What kind of distribution is it? State ϕ as a function of δ. (3 P)
The random variable is obviously Laplace distributed. Thus applies:

$$\phi = \frac{1}{2\delta}.$$

(j) Sketch the distribution function $F_a(a)$ roughly. Then determine the distribution function as a function of ϕ. (6 P)



$$F_a(a) = \int_{-\infty}^{a} f_a(t)dt$$

here:

$$F_a(a) = \int\limits_{-\infty}^{a} \delta e^{-\frac{|t|}{\phi}} dt$$

Case distinction:

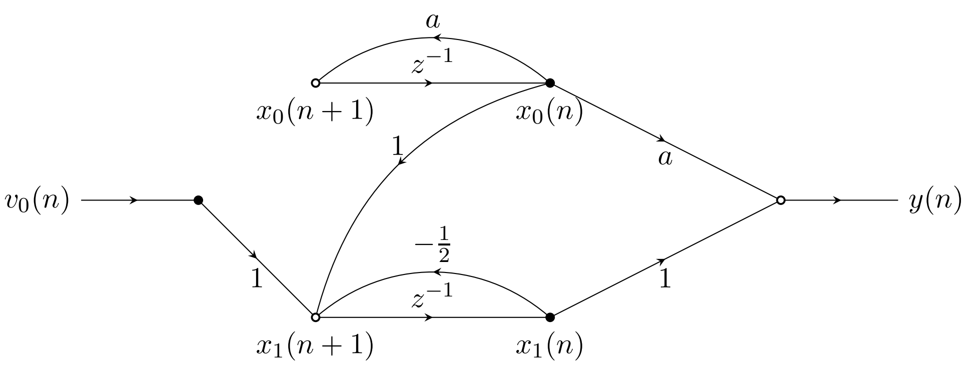
$$\begin{aligned} a \leq 0 : F_a(a) &= \int\limits_{-\infty}^{a} \delta e^{\frac{t}{\phi}} dt \\ &= \delta \left[\phi e^{\frac{t}{\phi}} \right]_{-\infty}^{a} \\ &= \delta \phi e^{\frac{a}{\phi}} \end{aligned}$$

$$\begin{aligned} a > 0 : F_a(a) &= \int\limits_{-\infty}^{0} \delta e^{\frac{t}{\phi}} dt + \int\limits_{0}^{a} \delta e^{\frac{-t}{\phi}} dt \\ &= \delta \phi + \delta \left[-\phi e^{\frac{-t}{\phi}} \right]_{0}^{a} \\ &= \delta \phi (2 - (e^{\frac{-a}{\phi}})) \end{aligned}$$

Task 2 (32.5 Points)

Part 1 *This part of the task can be solved independently of part 2.*

For the following part, let a system be given which is described by the following signal flow graph.



Furthermore, let the state space be described by the following equations:

$$\boldsymbol{x}(n+1) = \boldsymbol{A}\boldsymbol{x}(n) + \boldsymbol{B}\boldsymbol{v}(n)$$
$$\boldsymbol{y}(n) = \boldsymbol{C}\boldsymbol{x}(n) + \boldsymbol{D}\boldsymbol{v}(n)$$

(a) Specify the number of inputs L, states N and outputs R of the system. (1,5 P)
Inputs L: 1 States N: 2 Outputs R: 1

(b) Determine the matrices/vectors/scalars $\boldsymbol{A}$,$\boldsymbol{B}$,$\boldsymbol{C}$,$\boldsymbol{D}$ for the above system. (4 P)

$$\boldsymbol{A} = \begin{bmatrix} a & 0 \\ 1 & -\frac{1}{2} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} a & 1 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 0 \end{bmatrix}.$$

(c) Determine the characteristic polynomial of the system matrix $\boldsymbol{A}$. (3 P)
The characteristic polynomial is given by $N(z) = \det[z\boldsymbol{I} - \boldsymbol{A}]$. Then we have:

$$\begin{aligned} N(z) &= \det(z\boldsymbol{I} - \boldsymbol{A}) \\ &= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} a & 0 \\ 1 & -\frac{1}{2} \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} z-a & 0 \\ -1 & z+\frac{1}{2} \end{bmatrix}\right) \\ &= (z-a)\left(z+\frac{1}{2}\right) \\ &= z^2 + z\left(\frac{1}{2} - a\right) - \frac{a}{2} \end{aligned}$$

(d) Define an appropriate range of values for the parameter a so that the system is stable. (2 P)

The system is stable if all eigenvalues of the system matrix $\boldsymbol{A}$ lie within the unit circle.

The characteristic polynomial of the system is $N(z) = z^2 + z\left(\frac{1}{2} - a\right) - \frac{a}{2}$, whose eigenvalues are the solutions $z_1 = a$ and $z_2 = -\frac{1}{2}$. Thus, the system is stable if $a \in \mathbb{C}$ and $|a| < 1$.

If the system is initialised with zeros, the value a has no effect on the output signal $y(n)$, since the state $x_0(n)$ is not dependent on the input signal $v_0(n)$ and therefore always has the value 0.

(e) Determine the transfer function $H(z)$ of the system. (6 P)

The transfer function $H(z)$ is given by $H(z) = \boldsymbol{C}\left[z\boldsymbol{I} - \boldsymbol{A}\right]^{-1}\boldsymbol{B} + \boldsymbol{D}$. Then we have:

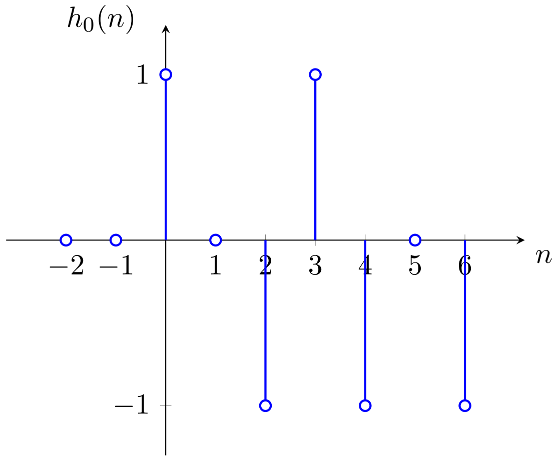
$$
\begin{aligned}
H(z) &= \begin{bmatrix} a & 1 \end{bmatrix} \left(z\boldsymbol{I} - \begin{bmatrix} a & 0 \\ 1 & -\frac{1}{2} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \\
&= \begin{bmatrix} a & 1 \end{bmatrix} \begin{bmatrix} z-a & 0 \\ -1 & z+\frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{z^2 + z\left(\frac{1}{2} - a\right) - \frac{a}{2}} \begin{bmatrix} a & 1 \end{bmatrix} \begin{bmatrix} z+\frac{1}{2} & 0 \\ 1 & z-a \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{z^2 + z\left(\frac{1}{2} - a\right) - \frac{a}{2}} \begin{bmatrix} a(z+\frac{1}{2}) + 1 & z-a \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{z^2 + z\left(\frac{1}{2} - a\right) - \frac{a}{2}} (z-a) \\
&= \frac{(z-a)}{(z-a)(z+\frac{1}{2})} \\
&= \frac{1}{z+\frac{1}{2}}
\end{aligned}
$$

(f) Determine the impulse response $h_0(n)$. (2 P)

$$
\begin{aligned}
h_0(n) &= \mathcal{Z}^{-1}\{H(z)\} \\
&= \mathcal{Z}^{-1}\left\{\frac{1}{z+\frac{1}{2}}\right\} \\
&= \left(-\frac{1}{2}\right)^{n+1} \gamma_{-1}(n-1)
\end{aligned}
$$

Part 2 *This part of the task can be solved independently of part 1.*

Given the following discrete impulse response $h_0(n)$:



Whereby, in addition, the following applies:

$$h_0(n) = 0 \;\forall\, n < 0,$$
$$h_0(n) = -\frac{1}{2^{n-6}} \;\forall\, n > 6.$$

(g) Determine the equation of the impulse response based on weighted impulse and step sequences. (4 P)

$$h_0(n) = \gamma_0(n) - \gamma_0(n-2) + \gamma_0(n-3) - \gamma_0(n-4) - \frac{1}{2^{n-6}}\gamma_{-1}(n-6)$$

(h) Determine the transfer function $H(z)$. (4 P)

Using known correspondences:

$$\begin{aligned}
H(z) &= 1 - z^{-2} + z^{-3} - z^{-4} - \frac{z^{-5}}{z-\frac{1}{2}} \\
&= \frac{\left(z-\frac{1}{2}\right)\left(1 - z^{-2} + z^{-3} - z^{-4}\right) - z^{-5}}{z-\frac{1}{2}} \\
&= \frac{z - \frac{1}{2} - z^{-1} + \frac{3}{2}z^{-2} - \frac{3}{2}z^{-3} + \frac{1}{2}z^{-4} - z^{-5}}{z-\frac{1}{2}} \\
&= \frac{1 - \frac{1}{2}z^{-1} - z^{-2} + \frac{3}{2}z^{-3} - \frac{3}{2}z^{-4} + \frac{1}{2}z^{-5} - z^{-6}}{1-\frac{1}{2}z^{-1}}
\end{aligned}$$

(i) Determine the difference equation. (4 P)

From the transfer function $H(z)$

$$H(z) = \frac{1 - \frac{1}{2}z^{-1} - z^{-2} + \frac{3}{2}z^{-3} - \frac{3}{2}z^{-4} + \frac{1}{2}z^{-5} - z^{-6}}{1 - \frac{1}{2}z^{-1}}$$

$$\frac{Y(z)}{V(z)} = \frac{1 - \frac{1}{2}z^{-1} - z^{-2} + \frac{3}{2}z^{-3} - \frac{3}{2}z^{-4} + \frac{1}{2}z^{-5} - z^{-6}}{1 - \frac{1}{2}z^{-1}}$$

follows

$$\begin{aligned}
(1 - \frac{1}{2}z^{-1})Y(z) &= \left(1 - \frac{1}{2}z^{-1} - z^{-2} + \frac{3}{2}z^{-3} - \frac{3}{2}z^{-4} + \frac{1}{2}z^{-5} - z^{-6}\right) V(z) \\
Y(z) - \frac{1}{2}z^{-1}Y(z) &= \left(1 - \frac{1}{2}z^{-1} - z^{-2} + \frac{3}{2}z^{-3} - \frac{3}{2}z^{-4} + \frac{1}{2}z^{-5} - z^{-6}\right) V(z) \\
Y(z) &= \left(1 - \frac{1}{2}z^{-1} - z^{-2} + \frac{3}{2}z^{-3} - \frac{3}{2}z^{-4} + \frac{1}{2}z^{-5} - z^{-6}\right) V(z) + \frac{1}{2}z^{-1}Y(z).
\end{aligned}$$

Transformation into the time domain yields the difference equation:

$$\begin{aligned}
y(n) = v(n) - \frac{1}{2}v(n-1) - v(n-2) + \frac{3}{2}v(n-3) - \frac{3}{2}v(n-4) + \frac{1}{2}v(n-5) \\
- v(n-6) + \frac{1}{2}y(n-1)
\end{aligned}$$

(j) Does the system have a direct pass-through? Give reasons for your answer. (2 P)

The system has a pass-through. Possible reasons:

- The following applies to the drawn impulse response $h_0(0) = 1$.
- The determined impulse response $h_0(n)$ has an entry for $n = 0$.
- For the given transfer function $H(z)$ the following applies: Numerator degree = denominator degree.
- The definite difference equation $y(n)$ has an entry for $v(n = 0)$.

Task 3 (34 Points)

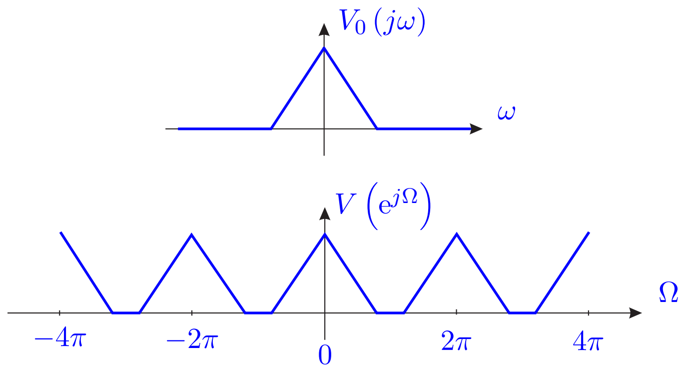
Part 1 *This part of the task can be solved independently of part 2 and part 3.*

(a) What steps are required to recover a signal modulated with a double-sideband modulation? Describe in complete sentences. (2 P)
First, the modulated signal is multiplied by the carrier signal. This causes a shift of the used spectrum into the baseband. A low-pass filter then removes the unwanted spectral components around twice the modulation frequency that have resulted from the multiplication.

(b) What purpose does the modulation of a signal serve, according to the lecture? What types of modulation do you know? Name at least three types of modulation. (2 P)
The purpose of modulation is to adapt the signal spectrum to the frequency range of the transmission, storage or processing medium to be used. Amplitude, phase and frequency modulation.

Part 2 *This part of the task can be solved independently of part 1 and part 3.*

(c) How does the Fourier transform $V_0\left(j\omega\right)$ of a continuous bandlimited signal $v_0(t)$ differ from the Fourier transform $V\left(e^{j\Omega}\right)$ of its with f_A sampled version $v_0\left(n\frac{1}{f_A}\right) = v(n)$ with $n \in \mathbb{Z}$. Describe in your own words. Make a sketch of the two spectra. Assume a real-valued spectrum of a real-valued signal. (5 P)
A sampling of the signal $v(t)$ with a sampling frequency f_A causes a periodic repetition of the spectrum with the period $\Omega_p = 2\pi$.

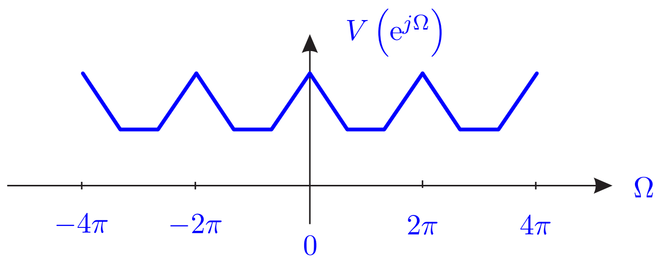


(d) What condition must the sampling frequency f_A meet so that no aliasing occurs? (2 P)
In order for aliasing not to occur, the following must apply to the sampling frequency f_A:

$$f_A \geq 2f_{max},$$

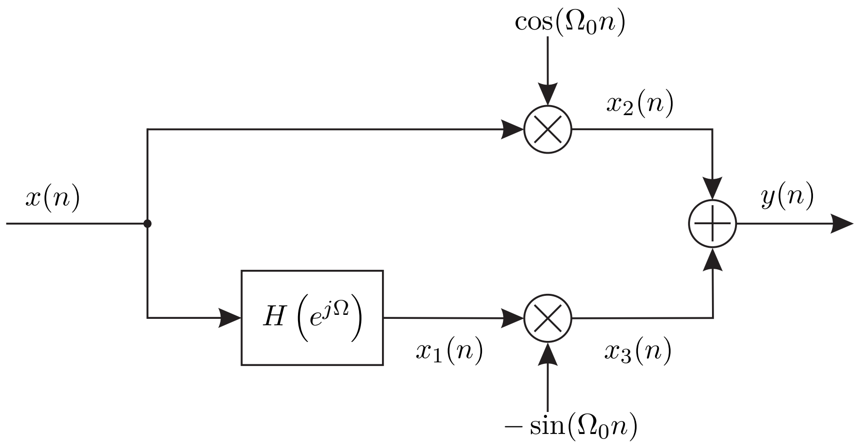
where f_{max} stands for the maximum signal frequency.

(e) How would the spectrum you drew from *(c)* look when aliased? Make a sketch. (3 P)

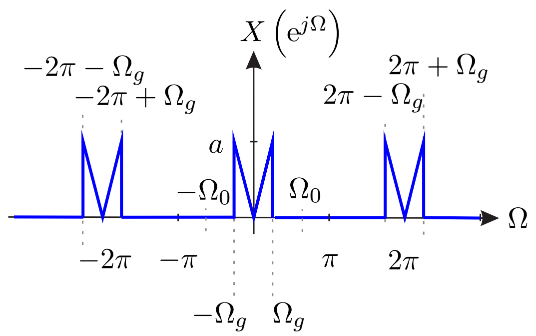


Part 3 *This part of the task can be solved independently of parts 1 and 2.*

Given the following system



where $H\left(e^{j\Omega}\right)$ is the discrete-time, zero-phase Hilbert transformer. The input signal $x(n)$ has the following 2π-periodic real spectrum:

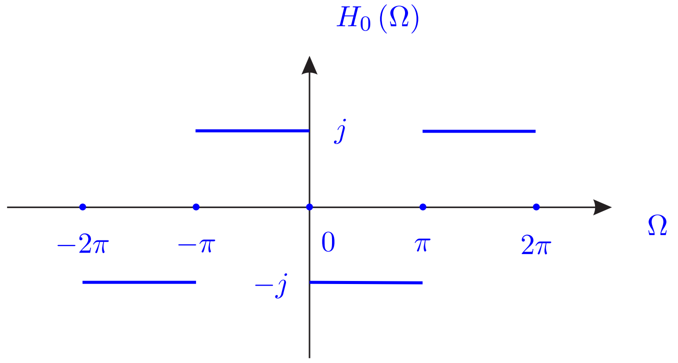


(f) What is the circuit given above often used for in practice? (2 P)
The circuit can be used to realise single-sideband modulation.

(g) Give the general definition of the discrete-time Hilbert transform and sketch its frequency response in the interval $-2\pi < \Omega < 2\pi$. Label all axes. (4 P)

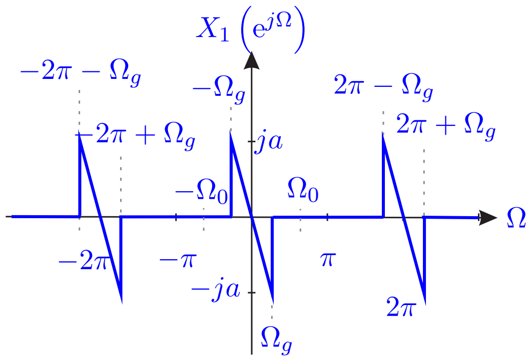
$$H\left(e^{j\Omega}\right) = H_0(\Omega)e^{-j\Omega n_0},$$
$$\text{mit } H_0(\Omega) = -j\,\text{sign}\left(\text{mod}(\Omega+\pi,2\pi)-\pi\right),$$

where n_0 stands for the time delay.



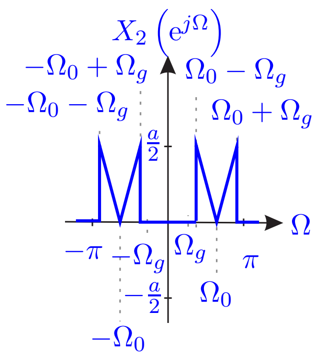
(h) Sketch the spectrum $X_1\left(e^{j\Omega}\right)$ of the signal $x_1(n)$ in the interval $-2\pi < \Omega < 2\pi$. (2 P)

The real part of $X_1\left(e^{j\Omega}\right)$ is zero, since zero phase.



(i) State the spectrum $X_2\left(e^{j\Omega}\right)$ of the signal $x_2(n)$ as a function of $X\left(e^{j\Omega}\right)$ and sketch it in the interval $-\pi < \Omega < \pi$. Assume that the system does not create an alias. (4 P)

$$X_2\left(e^{j\Omega}\right) = \frac{1}{2}\left[X\left(e^{j(\Omega-\Omega_0)}\right) + X\left(e^{j(\Omega+\Omega_0)}\right)\right].$$

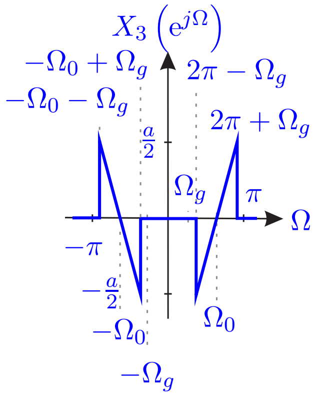


(j) State the spectrum $X_3\left(e^{j\Omega}\right)$ of the signal $x_3(n)$ as a function of $X\left(e^{j\Omega}\right)$ and sketch it in the interval $-\pi < \Omega < \pi$. Continue to assume that the system does not create an alias. (5 P)

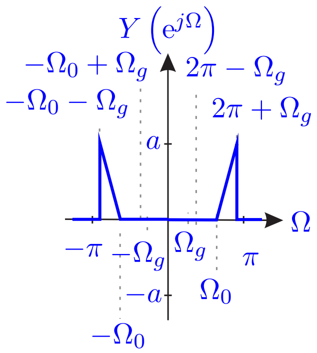
$$X_3\left(\mathrm{e}^{j\Omega}\right) = \frac{j}{2}\left[X_1\left(\mathrm{e}^{j(\Omega-\Omega_0)}\right) - X_1\left(\mathrm{e}^{j(\Omega+\Omega_0)}\right)\right],$$

$$= \frac{j}{2}\left[-X\left(\mathrm{e}^{j(\Omega-\Omega_0)}\right) j\,\mathrm{sign}\Big(\mathrm{mod}(\Omega-\Omega_0+\pi, 2\pi)-\pi\Big)\right.$$

$$\left. \ldots + X\left(\mathrm{e}^{j(\Omega+\Omega_0)}\right) j\,\mathrm{sign}\Big(\mathrm{mod}(\Omega+\Omega_0+\pi, 2\pi)-\pi\Big)\right].$$



(k) Sketch the spectrum $Y\left(e^{j\Omega}\right)$ of the signal $y(n)$ in the interval $-\pi < \Omega < \pi$. (3 P)



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